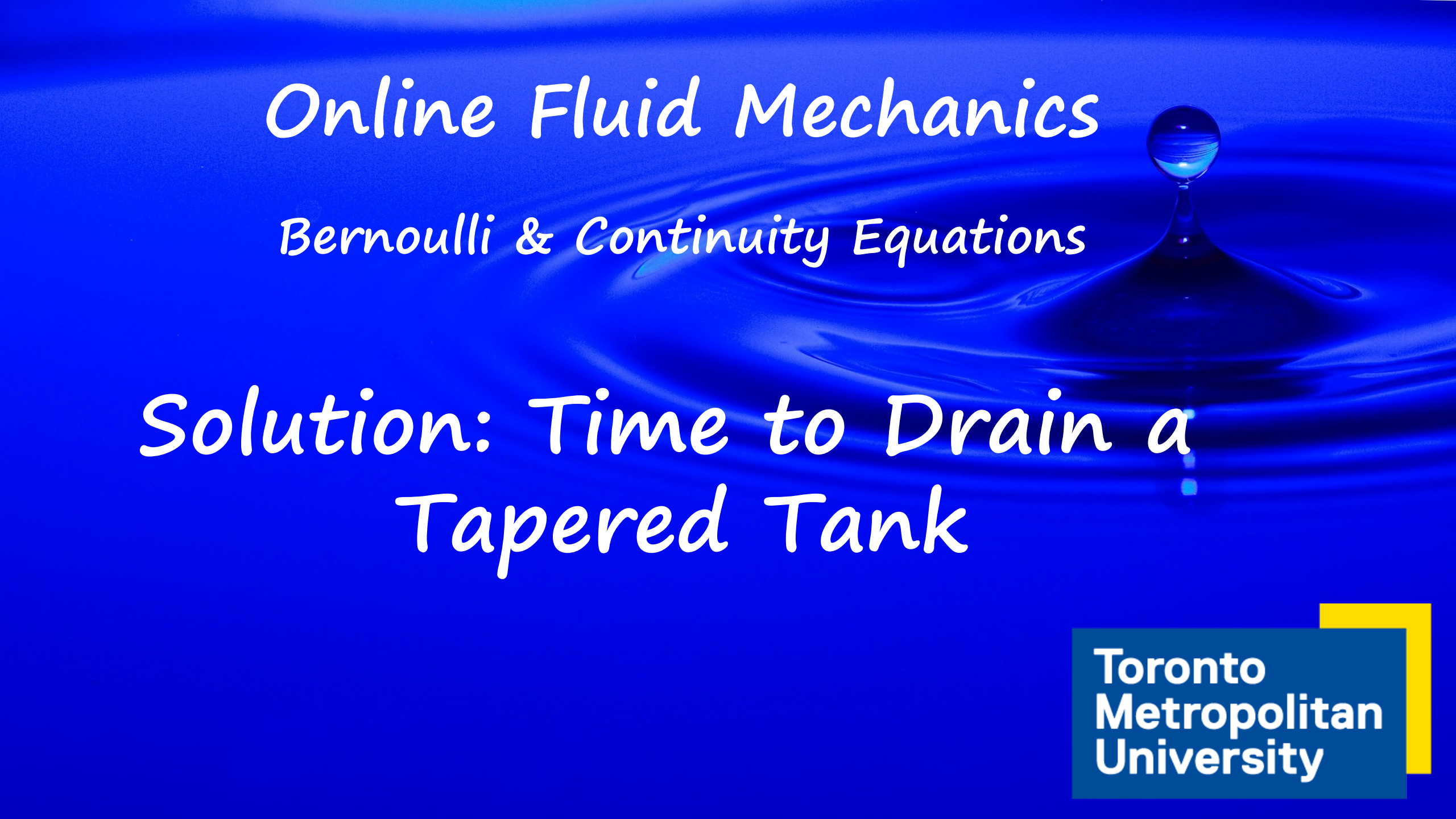
Online Fluid Mechanics

Bernoulli & Continuity Equations

Solution: Time to Drain a Tapered Tank



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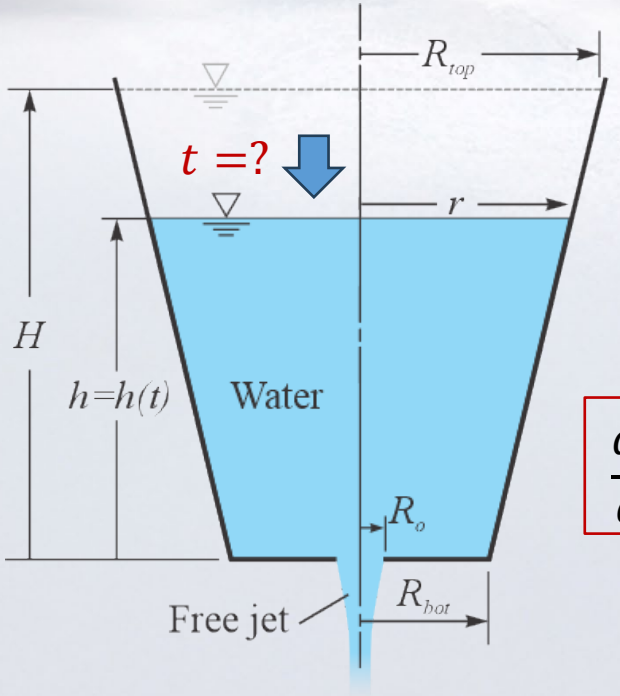
Fluid Mechanics Puzzle:

Consider two draining tanks:

- Same total volume of water
- Same starting depth
- ***Only*** difference is the taper

Which Tank Will Empty First?

☐ Tank A
☐ Tank B
☐ Both Same



$$\frac{dh}{dt} = -\frac{R_o^2}{r^2} C_D \sqrt{2gh}$$

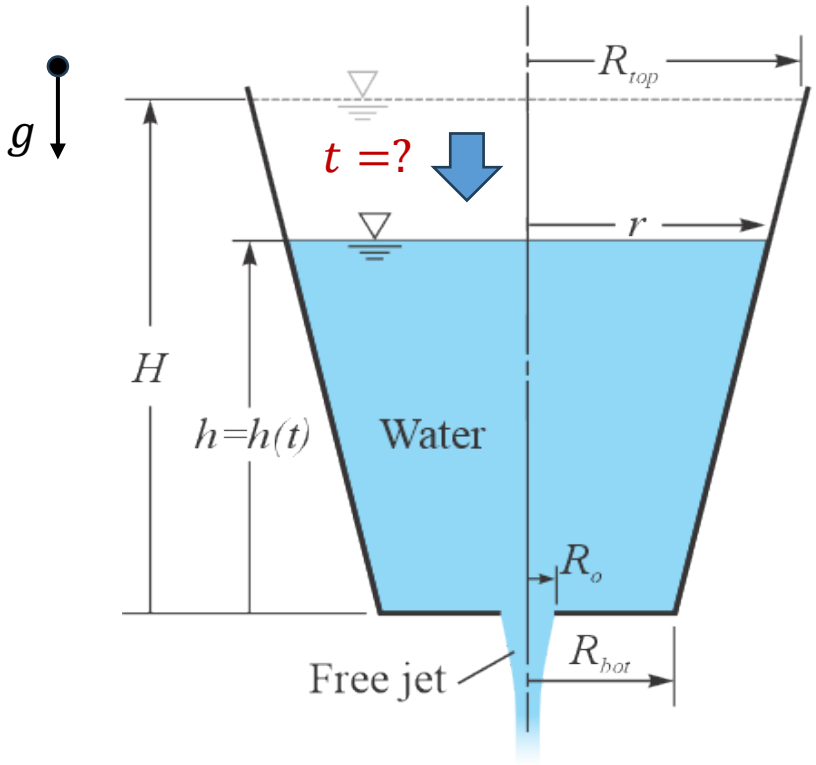
Result!

Time (t) for liquid level to fall from H to h:

$h = 0$ for time to fully drain

$$t = \frac{1}{R_o^2 C_D \sqrt{2g}} \left(2R_{bot}^2 (H^{\frac{1}{2}} - h^{\frac{1}{2}}) + \frac{4}{3} \beta R_{bot} (H^{\frac{3}{2}} - h^{\frac{3}{2}}) + \frac{2}{5} \beta^2 (H^{\frac{5}{2}} - h^{\frac{5}{2}}) \right) \quad \text{Eq. (1)}$$

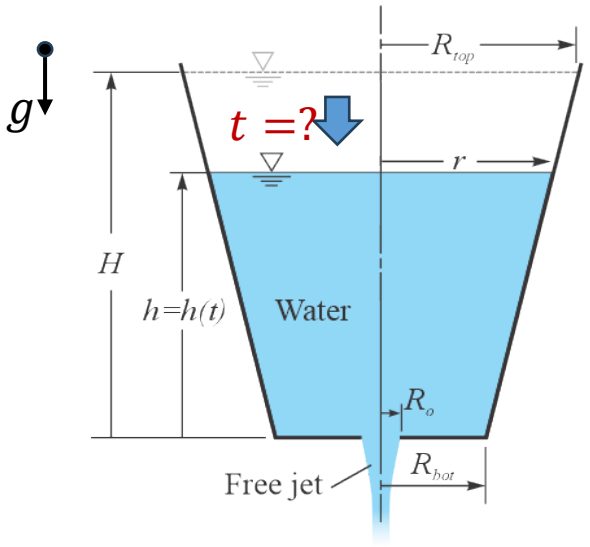
where the amount of taper is: $\beta = \left(\frac{R_{top} - R_{bot}}{H} \right)$



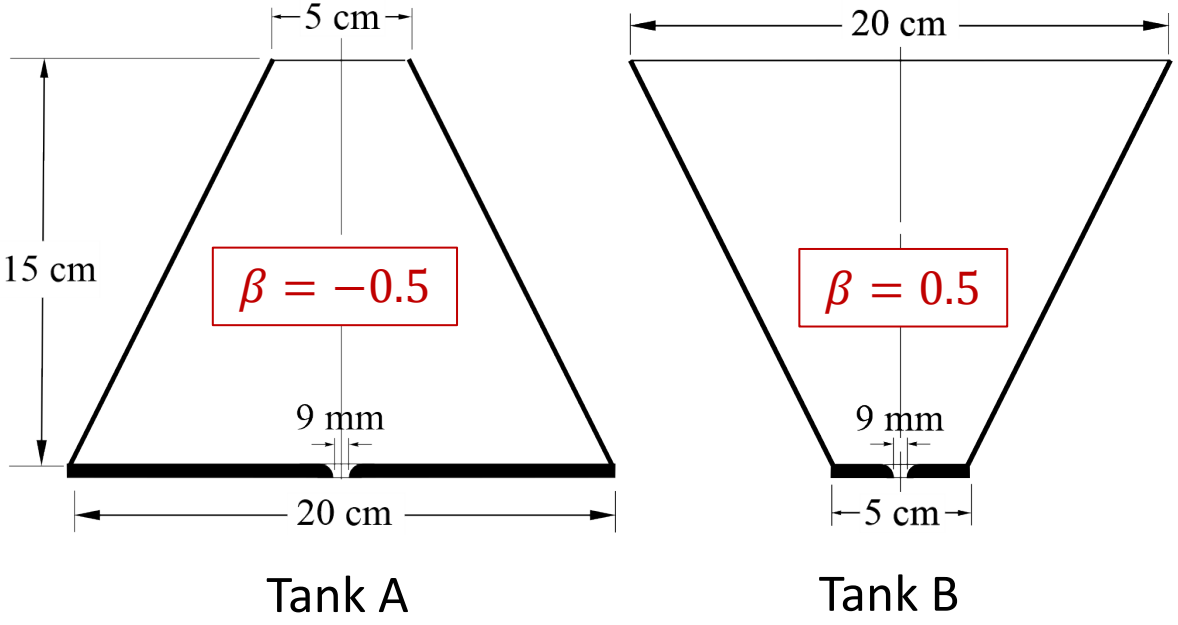
Final Equation!

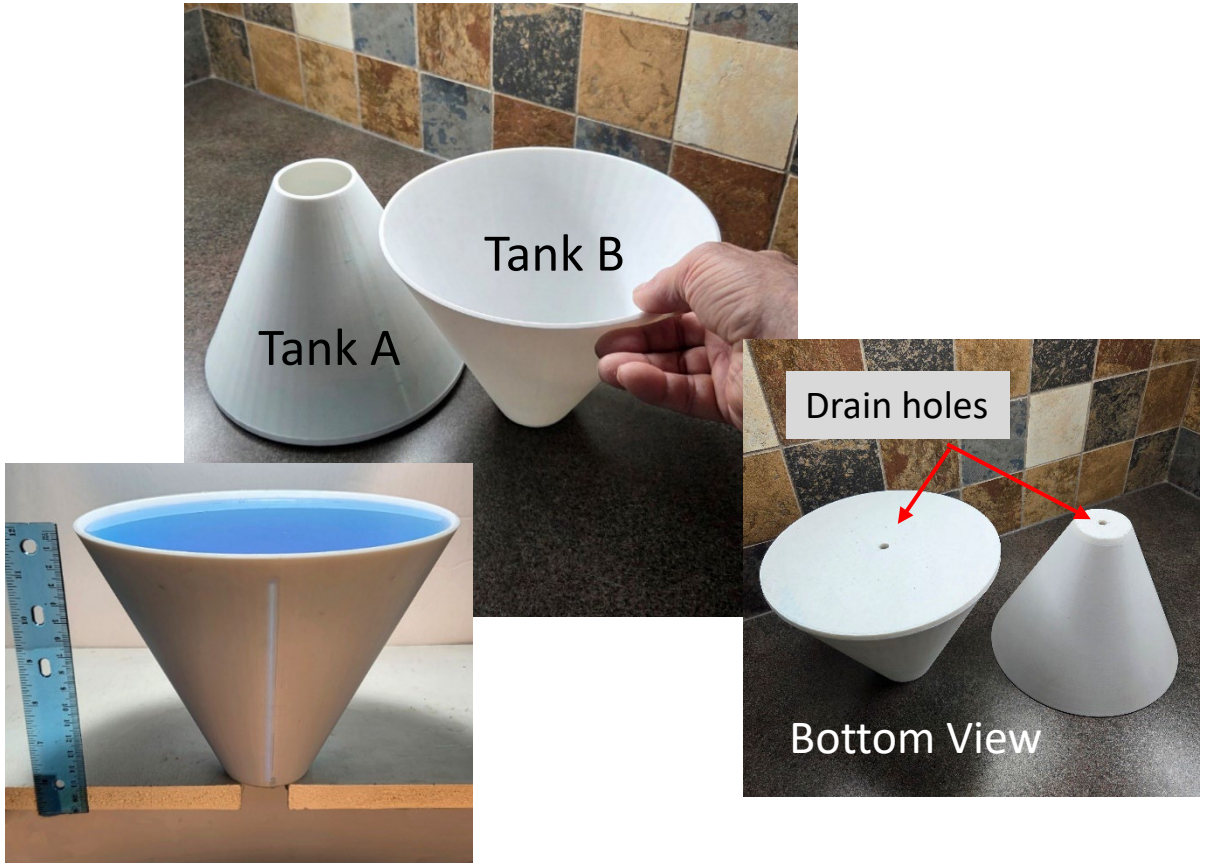
Time (t) for liquid level to fall from H to h:

$$t = \frac{1}{R_o^2 C_D \sqrt{2g}}\left(2R_{bot}^2(H^{\frac{1}{2}} - h^{\frac{1}{2}}) + \frac{4}{3}\beta R_{bot}(H^{\frac{3}{2}} - h^{\frac{3}{2}}) + \frac{2}{5}\beta^2(H^{\frac{5}{2}} - h^{\frac{5}{2}})\right) \quad \text{Eq. (1)}$$

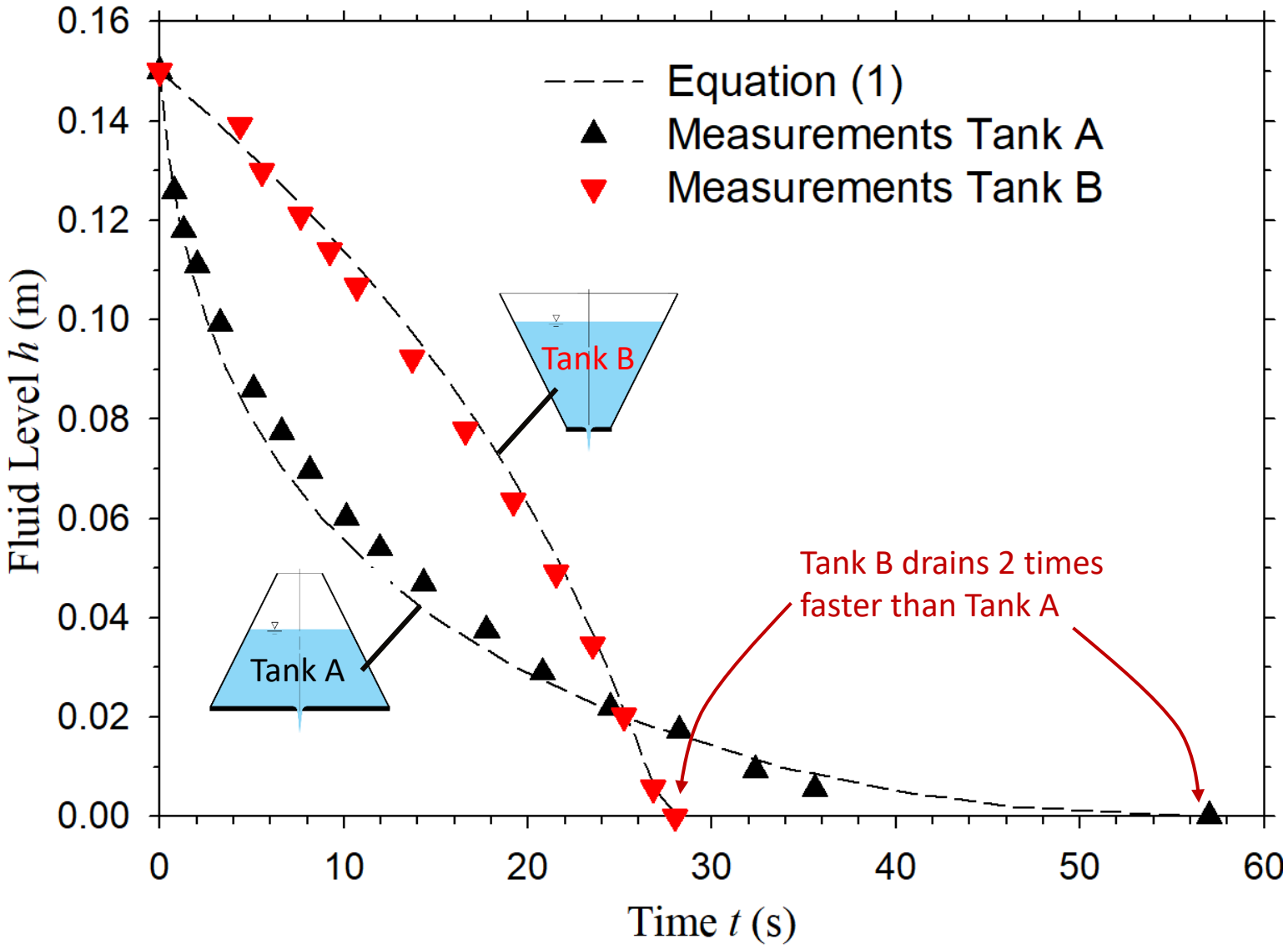


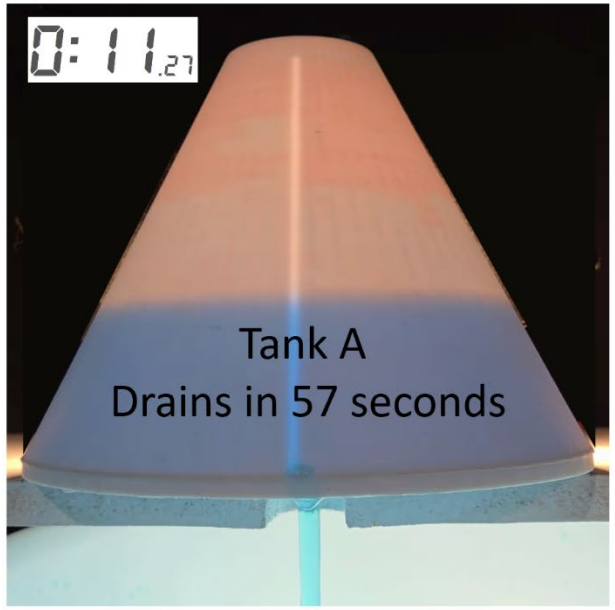
where the amount of taper is $\beta = \left(\frac{R_{top} - R_{bot}}{H}\right)$



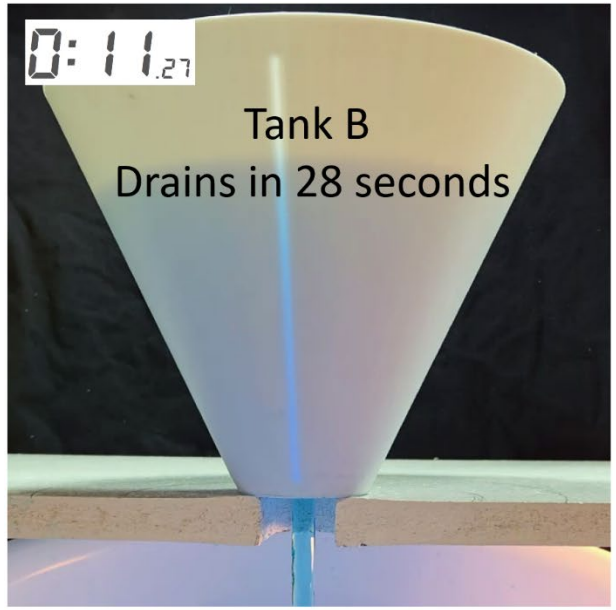


Comparison: Measurement & Theory





Tank A



Tank B

Solution

- Discharge velocity (modified Torricelli's law):

$$V_o = C_D\sqrt{2gh} \qquad (C_D = 1,\ \text{1-D inviscid flow})$$

- Instantaneous mass discharge rate:

$$\dot{m}_o = \rho V_o A_o = \rho V_o \pi R_o^2 = \rho\, C_D\sqrt{2gh}\,\pi R_o^2$$

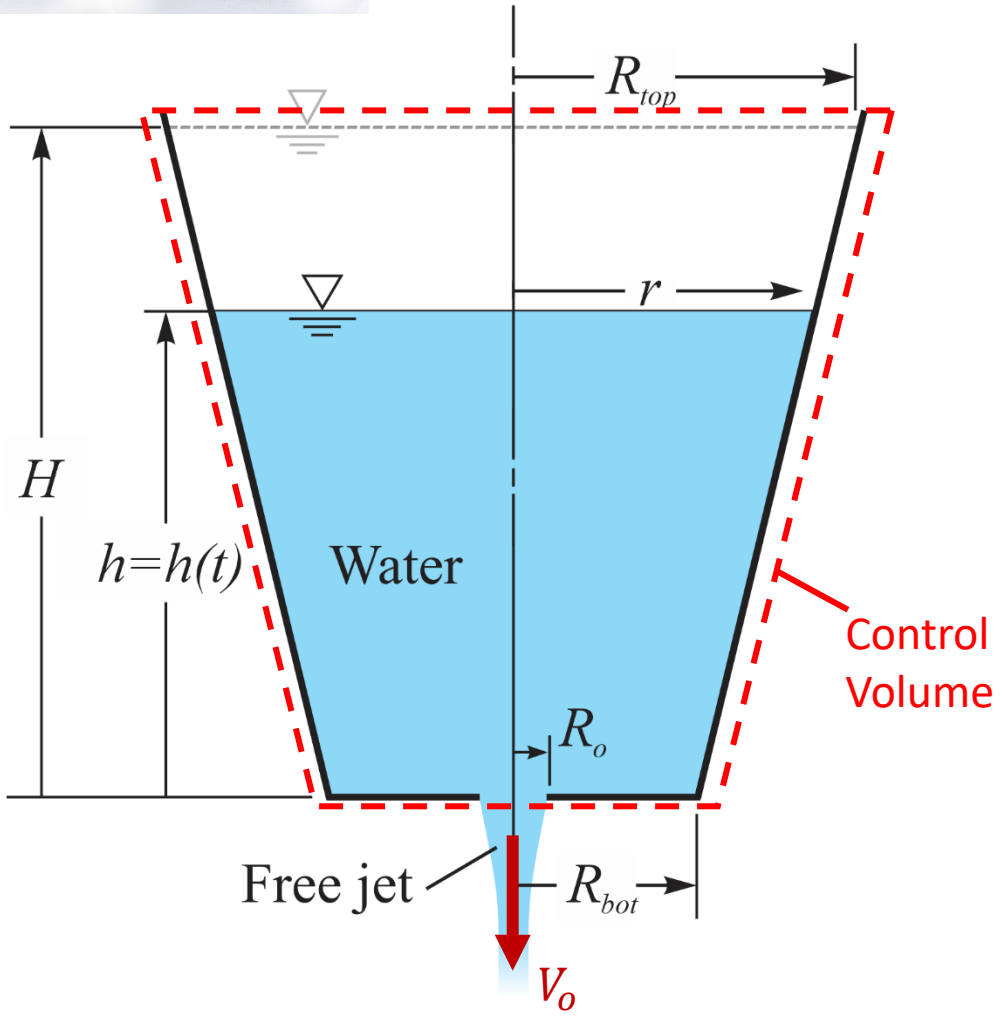
- Applying the continuity equation to the CV:

$$\frac{d}{dt}(m_{cv}) = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

0

$$\frac{d}{dt}(m_{cv}) = -\dot{m}_o = -\rho\, C_D\sqrt{2gh}\,\pi R_o^2$$

← Highest potential energy wins?



Solution

$$\frac{d}{dt}(m_{cv}) = -\rho C_D \sqrt{2gh}\,\pi R_o^2$$

- The rate of change mass in the control volume:

Constant ρ $d\forall = A dh$ $A = \pi r^2$

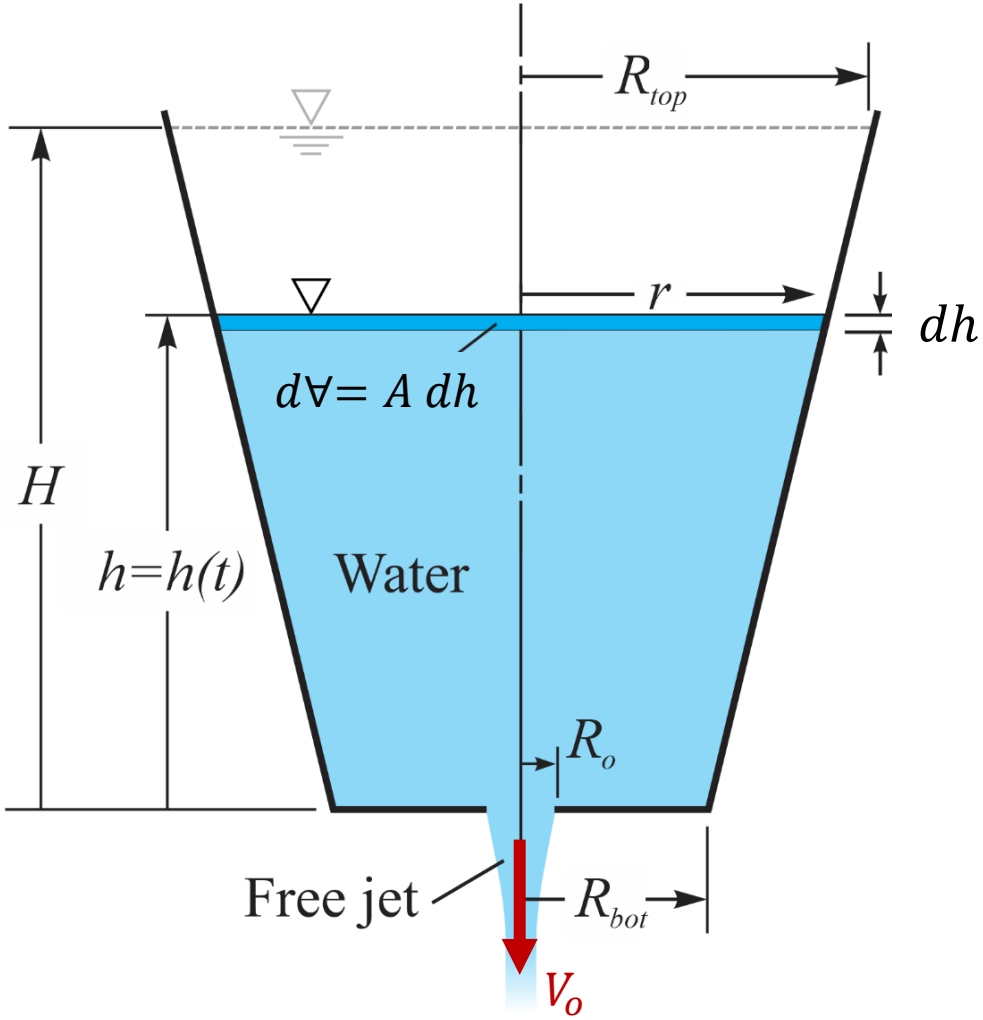
$$\frac{d}{dt}(m_{cv}) = \frac{d}{dt}(\rho\forall) = \rho\frac{d\forall}{dt} = \rho A\frac{dh}{dt}$$

- Making the substitution:

$$\rho\pi r^2\frac{dh}{dt} = -\rho C_D\sqrt{2gh}\,\pi R_o^2$$

$$\frac{dh}{dt} = -\frac{R_o^2}{r^2}C_D\sqrt{2gh}$$

$h(t)$ is decreasing with time, t



Solution

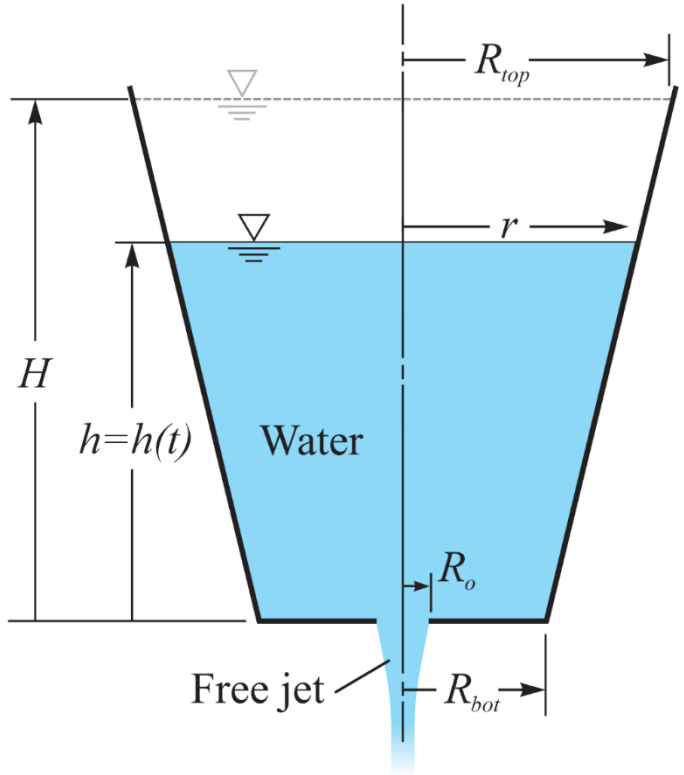
- Differential equation: $\frac{dh}{dt} = -\frac{R_o^2}{r^2} C_D \sqrt{2gh}$

$r = r(h)$

- The radius (r) varies linearly with height (h)

$$r = R_{bot} + \left(\frac{R_{top} - R_{bot}}{H}\right) h \quad \text{or} \quad r = R_{bot} + \beta h$$

where $\beta = \left(\frac{R_{top} - R_{bot}}{H}\right)$ ← "Taper Parameter"



- Differential equation becomes: $\frac{dh}{dt} = -\frac{R_o^2 C_D \sqrt{2gh}}{(R_{bot}+\beta h)^2} = -\frac{R_o^2 C_D \sqrt{2gh}}{R_{bot}^2 + 2\beta R_{bot} h + \beta^2 h^2}$

Solution

- Ordinary differential equation: $\frac{dh}{dt} = -\frac{R_o^2 C_D \sqrt{2gh}}{R_{bot}^2 + 2\beta R_{bot} h + \beta^2 h^2}$

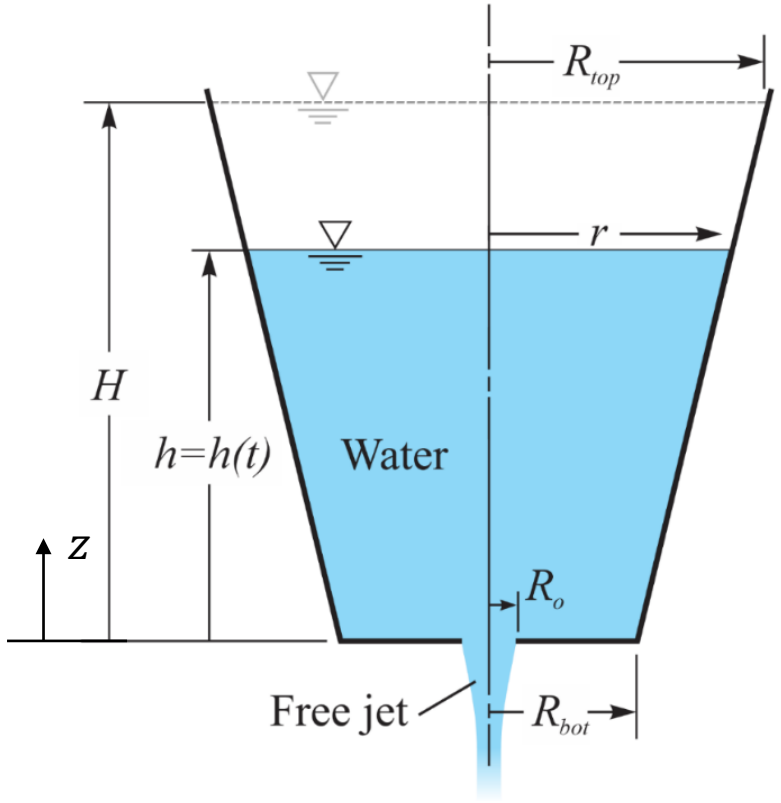
- Denoting z as the vertical coordinate. Water level is $z = h(t)$

- Separating variables:

$$\int_0^t dt = -\frac{1}{R_o^2 C_D \sqrt{2g}} \int_H^{h(t)} \left(R_{bot}^2 z^{-\frac{1}{2}} + 2\beta R_{bot} z^{\frac{1}{2}} + \beta^2 z^{\frac{3}{2}} \right) dz$$

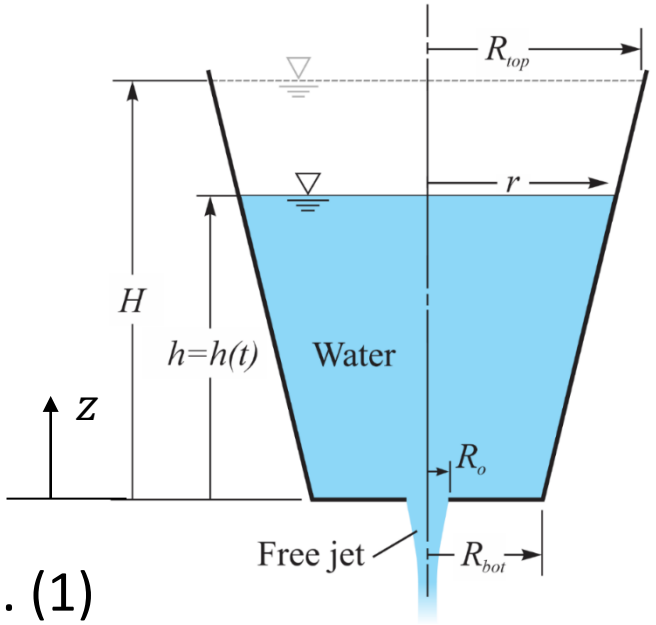
Integrating:

$$t = -\frac{1}{R_o^2 C_D \sqrt{2g}} \left[2R_{bot}^2 z^{\frac{1}{2}} + \frac{4}{3}\beta R_{bot} z^{\frac{3}{2}} + \frac{2}{5}\beta^2 z^{\frac{5}{2}} \right]_H^h$$



Solution

$$t = -\frac{1}{R_o^2 C_D \sqrt{2g}} \left[2R_{bot}^2 z^{\frac{1}{2}} + \frac{4}{3}\beta R_{bot} z^{\frac{3}{2}} + \frac{2}{5}\beta^2 z^{\frac{5}{2}} \right]_H^h$$



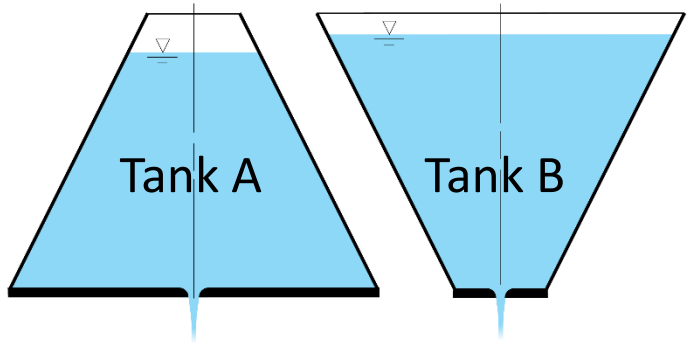
- Applying limits, the time for the tank to drain from H to h

$$t = \frac{1}{R_o^2 C_D \sqrt{2g}} \left(2R_{bot}^2 (H^{\frac{1}{2}} - h^{\frac{1}{2}}) + \frac{4}{3}\beta R_{bot} (H^{\frac{3}{2}} - h^{\frac{3}{2}}) + \frac{2}{5}\beta^2 (H^{\frac{5}{2}} - h^{\frac{5}{2}}) \right) \quad \text{Eq. (1)}$$

- Time to fully drain (t_o) is obtained by setting $h = 0$:

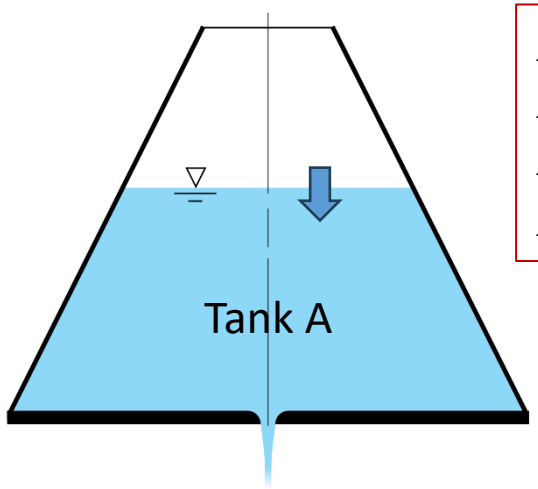
$$t_o = \frac{1}{R_o^2 C_D \sqrt{2g}} \left(2R_{bot}^2 H^{\frac{1}{2}} + \frac{4}{3}\beta R_{bot} H^{\frac{3}{2}} + \frac{2}{5}\beta^2 H^{\frac{5}{2}} \right) \quad \text{Eq. (2)}$$

- Now let's apply these results to our two 3D-printed tanks →



Solution

$$t_o = \frac{1}{R_o^2 C_D \sqrt{2g}}\left(2R_{bot}^2 H^{\frac{1}{2}} + \frac{4}{3}\beta R_{bot} H^{\frac{3}{2}} + \frac{2}{5}\beta^2 H^{\frac{5}{2}}\right) \qquad \text{Eq. (2)}$$



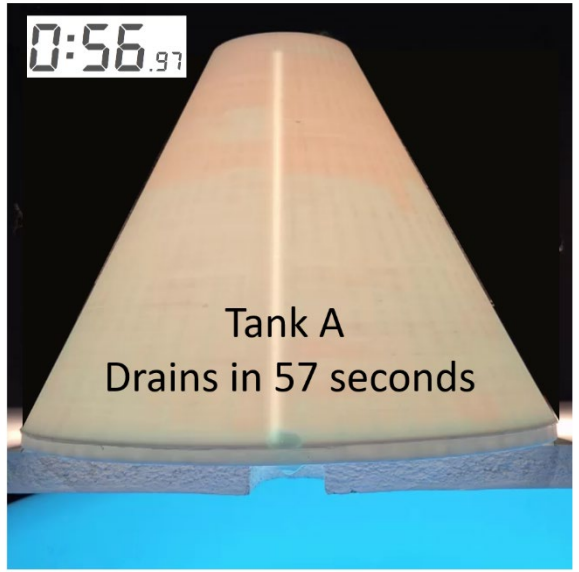
$H = 15\ cm$
$R_{bot} = 10\ cm$
$R_{top} = 2.5\ cm$
$R_o = 4.5\ mm$

- For Tank A: $\beta = \frac{R_{top} - R_{bot}}{H} = \frac{2.5cm - 10cm}{15cm} = -0.5$

$$t_o = \frac{1}{(0.0045m)^2\ (1.0)\sqrt{2(9.81\frac{m}{s^2})}}\left(2(0.1m)^2(0.15m)^{\frac{1}{2}} + \frac{4}{3}(-0.5)0.1m(0.15m)^{\frac{3}{2}} + \frac{2}{5}(-0.5)^2(0.15m)^{\frac{5}{2}}\right)$$

Inviscid flow

$t_o = 52.9\ s$ (inviscid flow, $C_D = 1$)

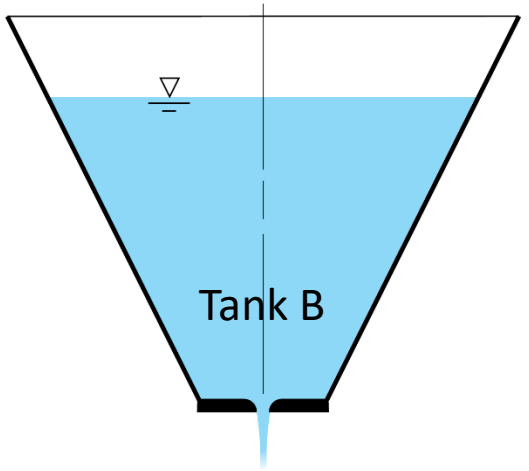


- Actual time for the tank to drain (viscous flow) was 57 seconds

- Average discharge coefficient: $C_D = \frac{t_{o,inviscid}}{t_{o,water}} = \frac{52.9\ s}{57\ s} = 0.93$

Solution

$$t_o = \frac{1}{R_o^2 C_D \sqrt{2g}}\left(2R_{bot}^2 H^{\frac{1}{2}} + \frac{4}{3}\beta R_{bot} H^{\frac{3}{2}} + \frac{2}{5}\beta^2 H^{\frac{5}{2}}\right) \quad \text{Eq. (2)}$$

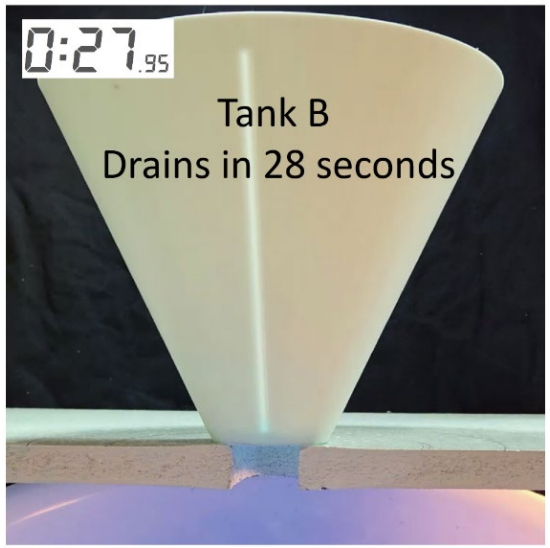


$H = 15\ cm$
$R_{bot} = 2.5\ cm$
$R_{top} = 10\ cm$
$R_o = 4.5\ mm$

- For Tank B: $\beta = \frac{R_{top} - R_{bot}}{H} = \frac{10\ cm - 2.5\ cm}{15\ cm} = +0.5$

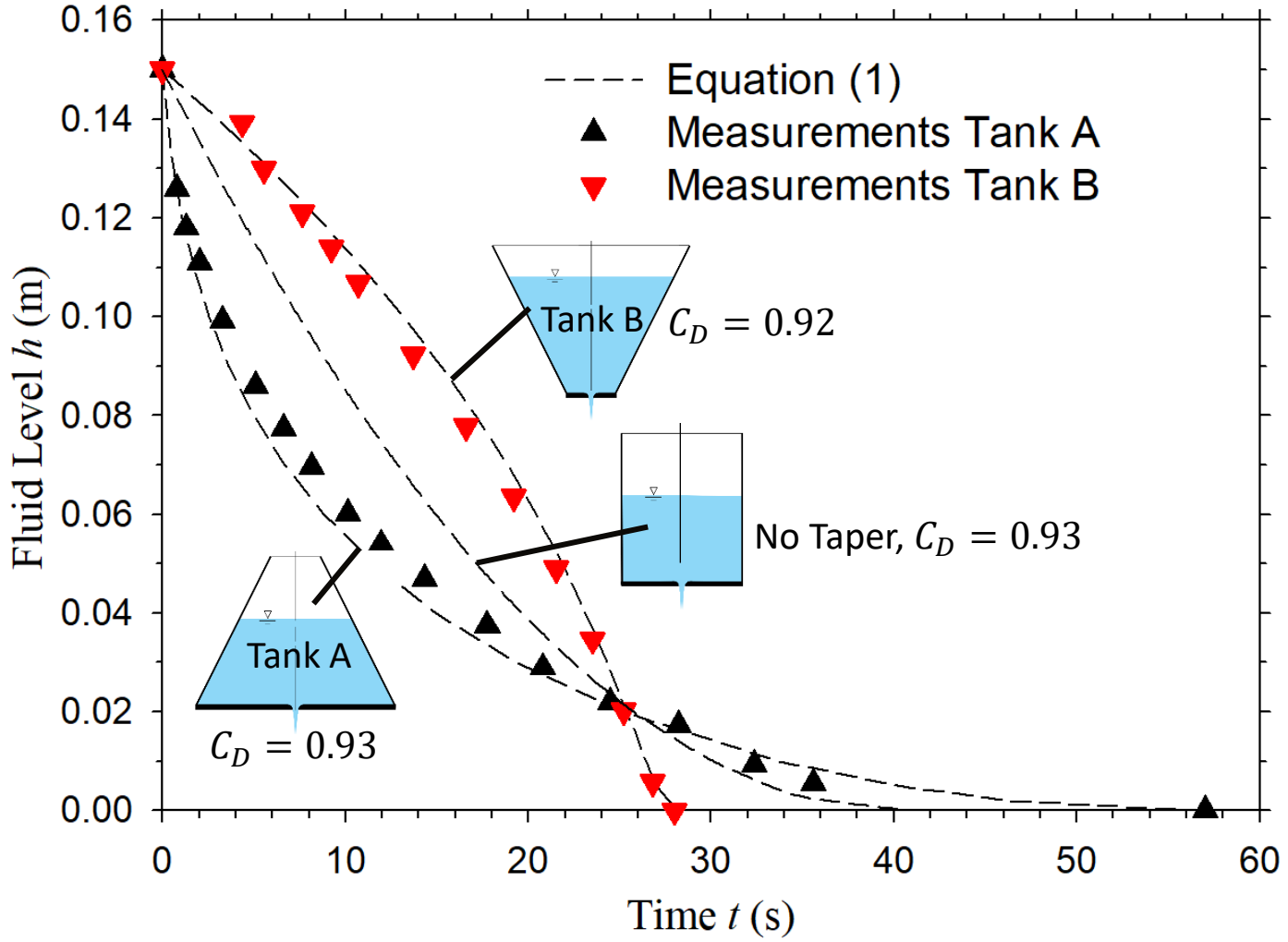
$$t_o = \frac{1}{(0.0045m)^2\ (1.0)\sqrt{2(9.81\frac{m}{s^2})}}\left(2(0.025m)^2(0.15m)^{\frac{1}{2}} + \frac{4}{3}(0.5)0.025m(0.15m)^{\frac{3}{2}} + \frac{2}{5}(0.5)^2(0.15m)^{\frac{5}{2}}\right)$$

$t_o = 25.9\ s$ (inviscid flow)



- Actual time for the Tank B to drain (viscous flow) was 28 seconds

- Average discharge coefficient: $C_D = \frac{t_{o,inviscid}}{t_{o,water}} = \frac{25.9\ s}{28\ s} = 0.92$



Limitations

- Quasi-steady flow ($A_o \ll A_{tank}$)
- Low viscous losses
- Complex flow as $h \to 0$
- Uncertainty in waterline location (PLA not ideal)

END NOTES

Presentation prepared and delivered by Professor David Naylor

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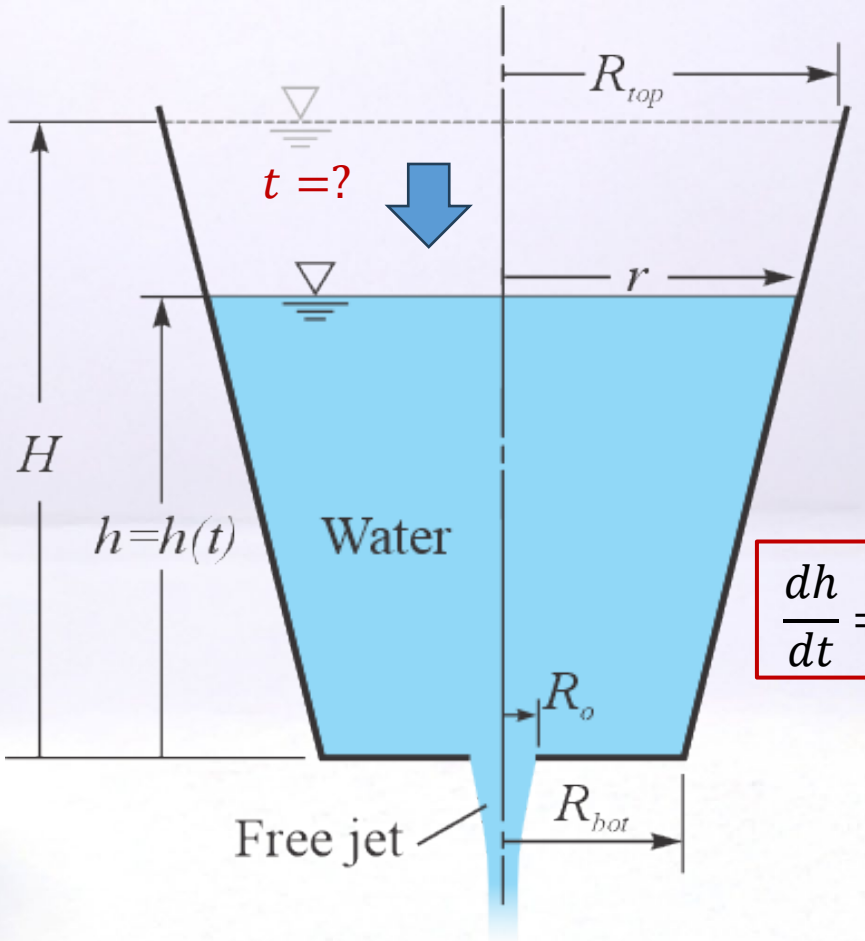
Fluid Mechanics Solution:

Toronto Metropolitan University

Which Tank Will Empty First?



$$\frac{dh}{dt} = -\frac{R_o^2}{r^2} C_D \sqrt{2gh}$$