MEC516/BME516: Fluid Mechanics I

Review: Final Exam Question

Navier-Stokes Solution Example

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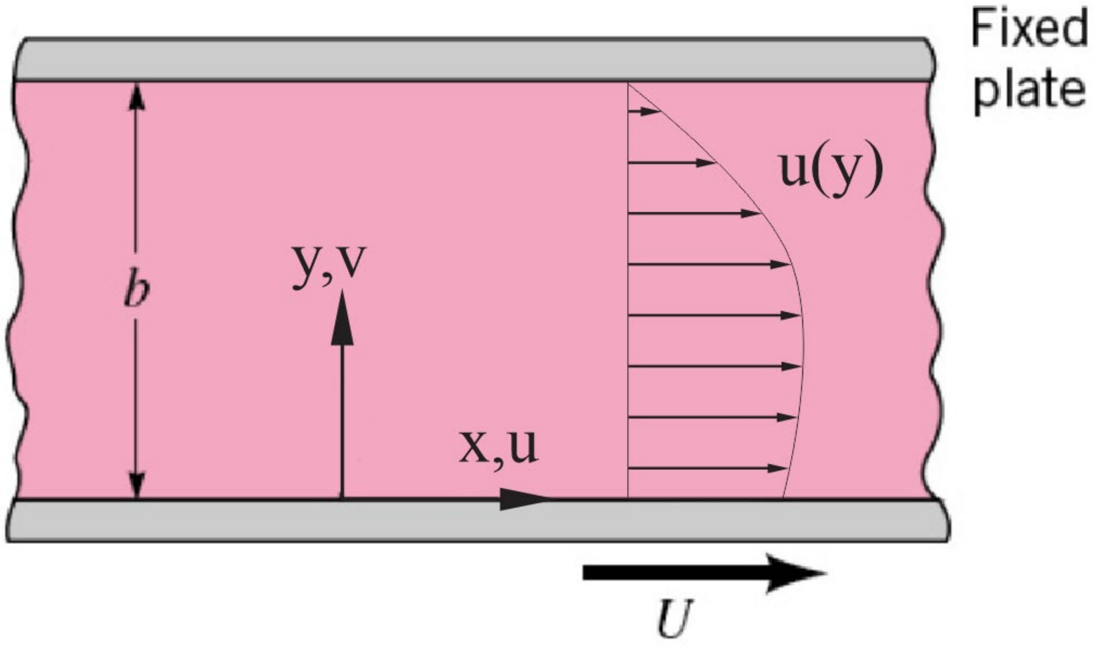
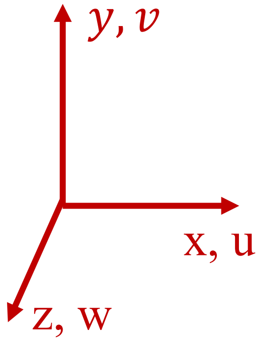
Example: Navier-Stokes Solution (Final Exam Fall 2014)

Consider steady laminar flow of a viscous incompressible fluid between two very long parallel plates with spacing b. Far from the channel entrance, the z- and y- components of fluid velocity are zero (v=w=0). The upper wall is stationary. The flow is driven by the motion of the lower wall at constant velocity U and by a pressure gradient in the x-direction, dp/dx. Effects of gravity can be neglected. Starting from the full incompressible continuity and x-momentum equations, derive an expression for fluid velocity, u(y).



(A combo of *Couette Flow* and *Poiseuille Flow* – two solutions done in lecture)

The Continuity Equation (Conservation of Mass)



- For compressible flow:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

- For incompressible flow ($\rho = const.$):

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

Valid for steady or unsteady incompressible flow.

The Navier-Stokes Equations (Conservation of Momentum)

For a Newtonian fluid with constant properties ($\rho = const, \mu = const$):

x-momentum:

$$\boxed{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x}$$

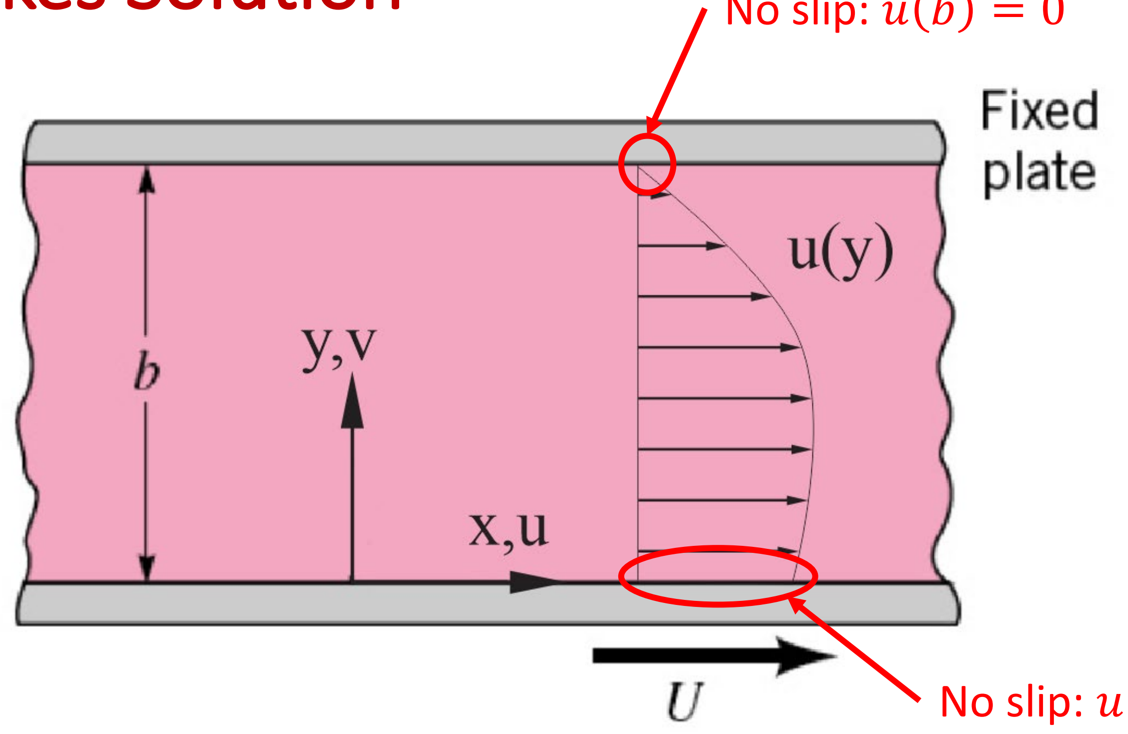
y-momentum:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y$$

z-momentum:

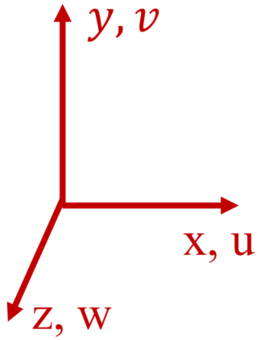
$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z$$

Example: Navier-Stokes Solution



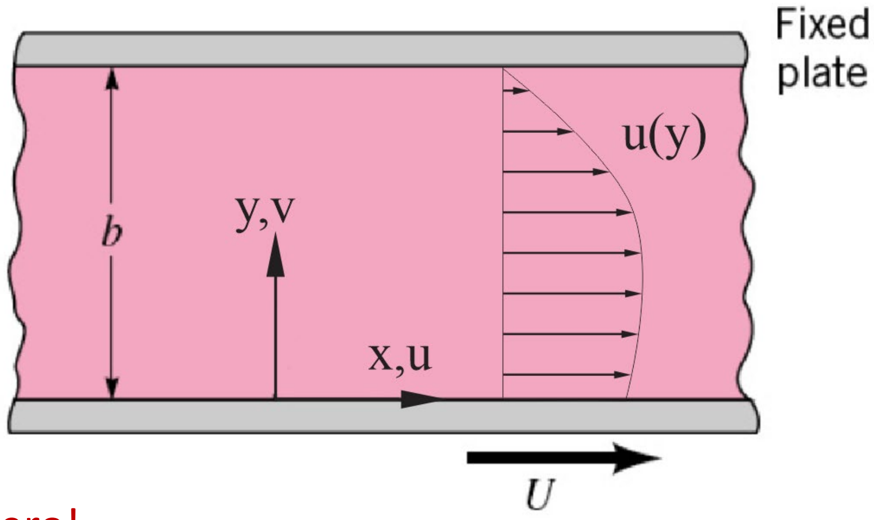
No slip: $u(b) = 0$

No slip: $u(0) = U$



- Note from problem statement:
  - No flow in the y- or z- directions: $v = w = 0$,
  - Flow driven by pressure gradient: $\frac{dp}{dx} \neq 0$
  - Effects of gravity can be neglected, $g_x = 0$
  - Boundary conditions at $y = 0, b$

Example: Navier-Stokes Solution



- Start from continuity equation, incompressible flow ($\rho = const.$):

$$\frac{\partial u}{\partial x} + \underset{v=0}{\cancel{\frac{\partial v}{\partial y}}} + \underset{w=0}{\cancel{\frac{\partial w}{\partial z}}} = 0 \quad \text{Thus:} \quad \frac{\partial u}{\partial x} = 0 \quad \text{Fully developed flow}$$

- Now consider x-momentum:

$$\rho\left(\underset{\substack{0\\ \text{steady}}}{\cancel{\frac{\partial u}{\partial t}}} + \underset{\text{Fully Dev.}}{\cancel{u\frac{\partial u}{\partial x}}} + \underset{v=0}{\cancel{v\frac{\partial u}{\partial y}}} + \underset{w=0}{\cancel{w\frac{\partial u}{\partial z}}}\right) = -\overset{\substack{\frac{dp}{dx}\\ \downarrow}}{\frac{\partial p}{\partial x}} + \mu\left(\underset{\text{Fully Dev.}}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \overset{\substack{\text{Not zero!}\\ \downarrow}}{\frac{\partial^2 u}{\partial y^2}} + \underset{0}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right) + \underset{0}{\cancel{\rho g_x}}$$

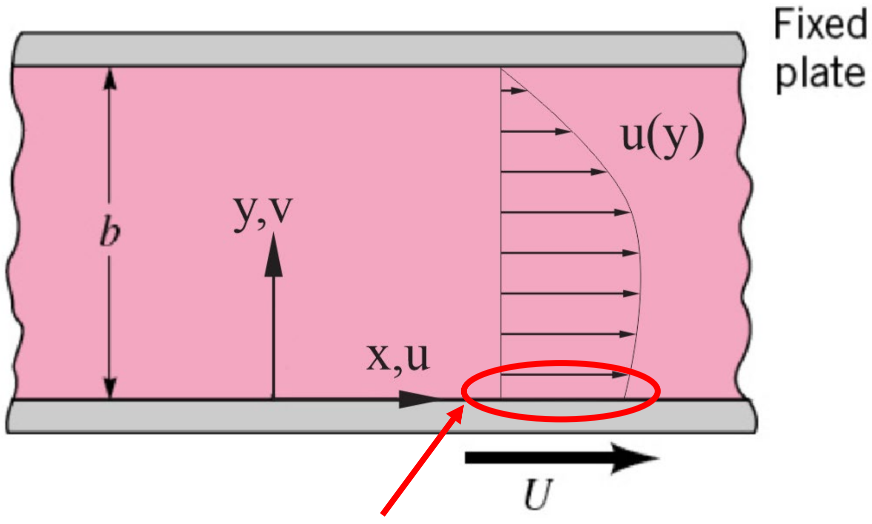
- Simplifying:

$$\left.\mu\frac{d^2 u}{dy^2} = \frac{dp}{dx} = const\ \right\}$$

p is only a function of x

u is only a function of y

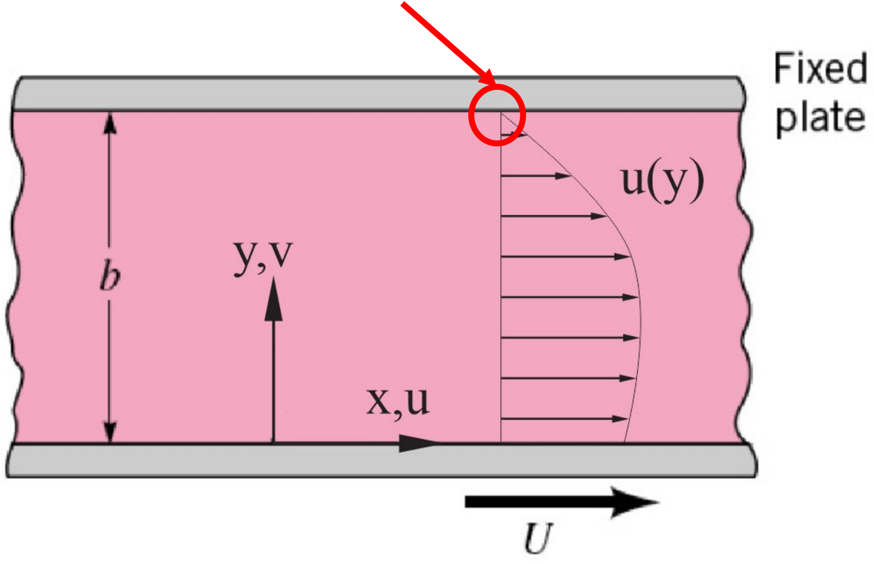
Example: Navier-Stokes Solution



No slip: $u(0) = U$

- Result: $$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = const$$
- Integrating with respect to y: $$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$
- Integrating again: $$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1y + C_2 \quad (*)$$
- The two constants (C_1, C_2) are found from the two velocity boundary conditions
- Substitute $y = 0, u = U$ into equation (*): $$\underset{\overset{\uparrow}{U}}{u} = \frac{1}{2\mu}\frac{dp}{dx}\underset{0}{\cancel{y}}^2 + C_1\underset{0}{\cancel{y}} + C_2$$
- Thus: $$C_2 = U$$

Example: Navier-Stokes Solution


No slip: $u(b) = 0$

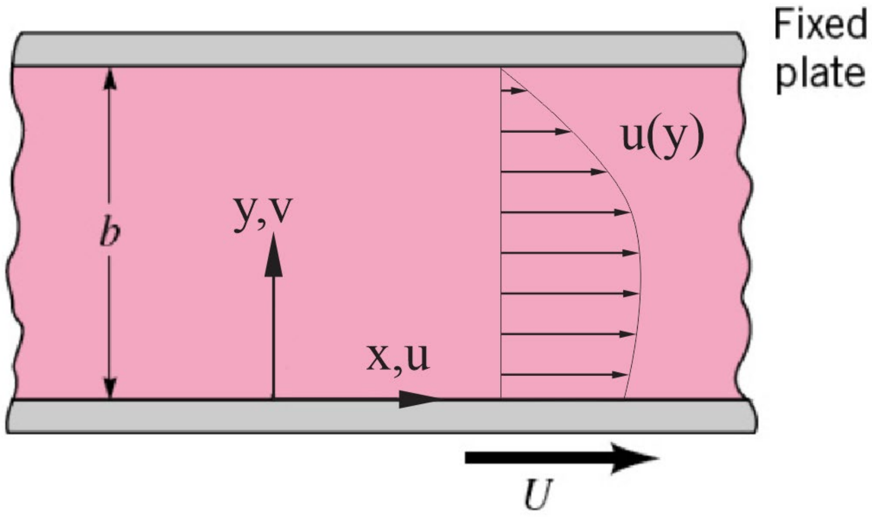
- Equation (*) becomes: $u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + U$ (*)

- Now we use the upper velocity boundary condition to get C_1

- Substitute $y = b, u = 0$ into equation (*): $$\underset{\searrow\, 0}{u} = \frac{1}{2\mu}\frac{dp}{dx}\underset{\underset{b^2}{\uparrow}}{y^2} + C_1 \underset{\underset{b}{\uparrow}}{y} + U$$

- Thus: $$0 = \frac{1}{2\mu}\frac{dp}{dx}b^2 + C_1 b + U \quad \rightarrow \quad C_1 = \frac{-b}{2\mu}\frac{dp}{dx} - \frac{U}{b}$$

Example: Navier-Stokes Solution



- We now have: $C_1 = \frac{-b}{2\mu}\frac{dp}{dx} - \frac{U}{b}$
- Sub. into Equation (*): $u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + U$
- Result: $u = \frac{1}{2\mu}\frac{dp}{dx}y^2 - \frac{b}{2\mu}\frac{dp}{dx}y - \frac{U}{b}y + U$
- Collecting terms:

$$\boxed{u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - by\right) + U(1 - \frac{y}{b})}$$

← Answer