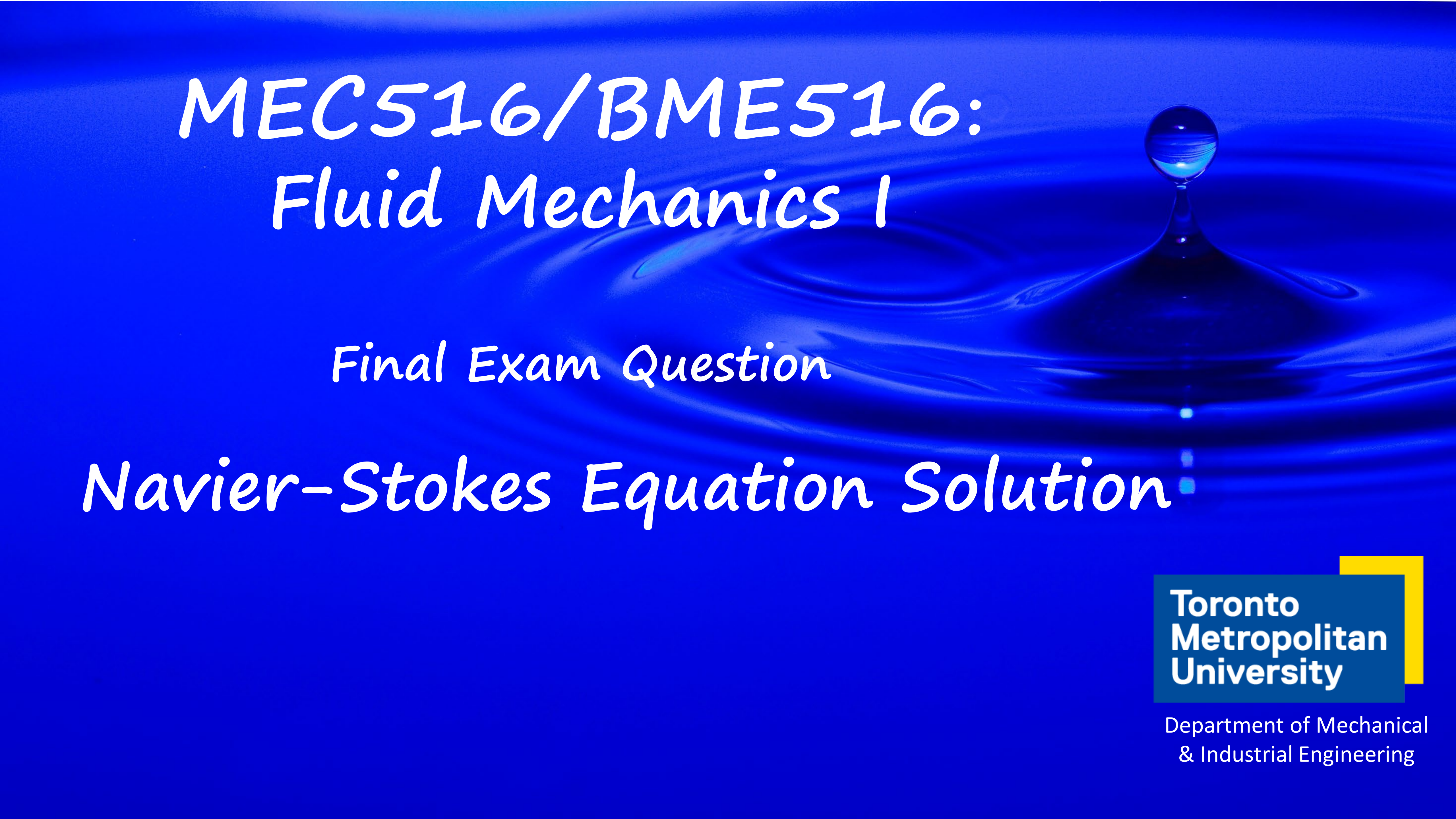


MEC516/BME516: Fluid Mechanics I

Final Exam Question

Navier-Stokes Equation Solution

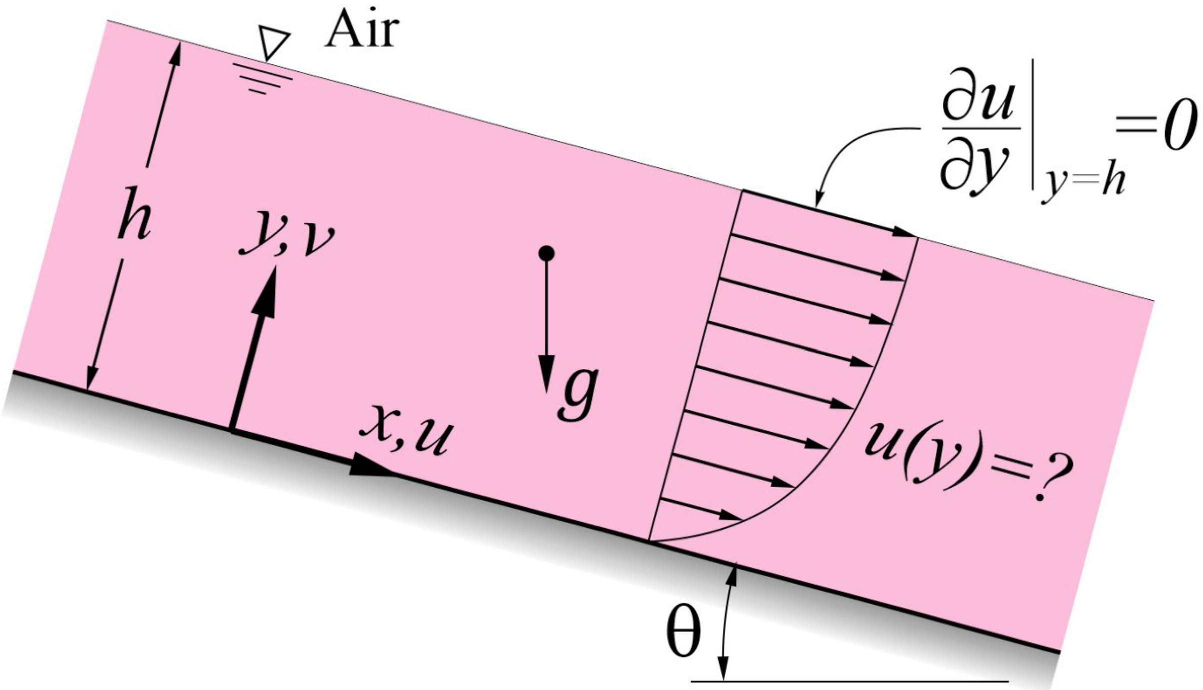
Toronto
Metropolitan
University

Department of Mechanical
& Industrial Engineering

Final Exam 2022: Navier-Stokes Equation Solution

A film of viscous liquid flows down a plate inclined at angle θ. The gravity-driven flow is steady, two-dimensional, incompressible and laminar. Far from the upper edge of the plate the film thickness (h) will be constant, and the y-component of fluid velocity is zero $(v = 0)$. There is no pressure gradient in the x-direction ($\partial p/\partial x=0$). The shear stress caused by the air at $y = h$ is negligible. Thus, the upper boundary condition is:

$$\left.\frac{\partial u}{\partial y}\right|_{y=h} = 0$$



Derive expressions for the:

(a) x-component of liquid velocity, $u(y)$

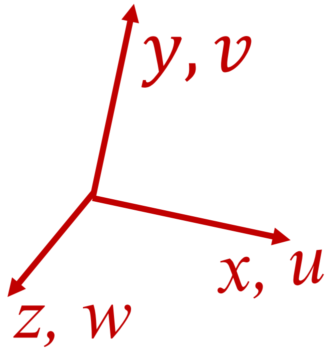
(b) Pressure gradient in the y-direction, $\frac{dp}{dy}$

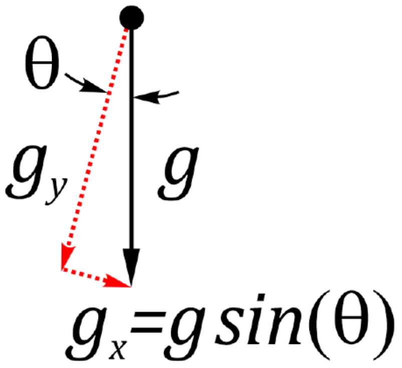
Navier-Stokes Equation Solution

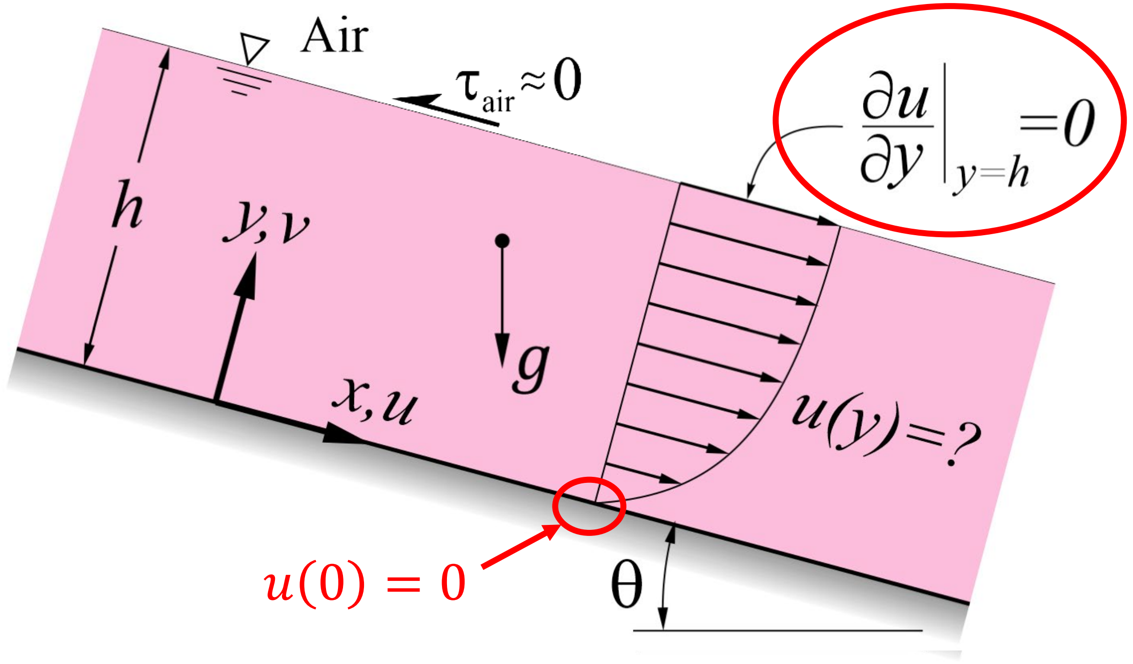
- From problem statement:
  - Two-dimensional: $w = 0$
  - No flow in the y-direction: $v = 0$
  - Gravity driven flow: $g_x = g\,\sin(\theta)$
  - No press. gradient parallel to plate: $\frac{\partial p}{\partial x} = 0$
  - Boundary conditions:
    - No-slip at $y = 0$
    - No shear stress (τ) at $y = h$

Shear stress at $y = h$:

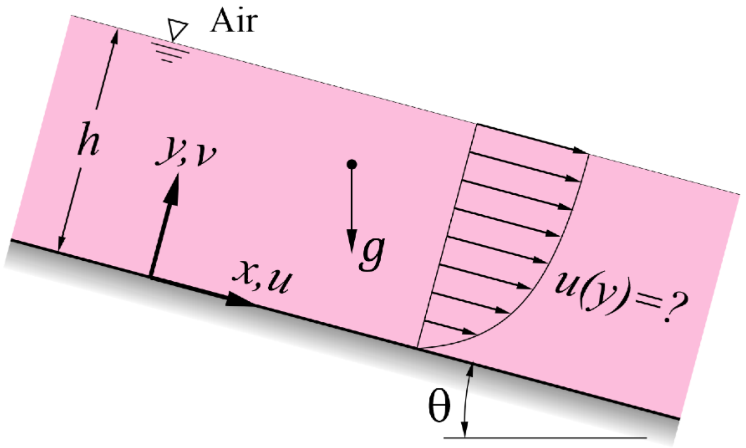
$$\tau = \mu \left.\frac{\partial u}{\partial y}\right|_{y=h} = 0$$







Navier-Stokes Equation Solution



(a) Start from continuity equation, incompressible flow ($\rho = const$):

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0 \quad \text{Thus:} \quad \frac{\partial u}{\partial x} = 0 \quad \text{Fully developed flow, } u = u(y)$$

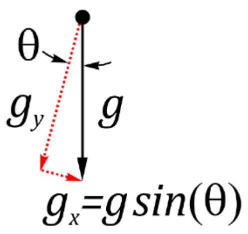
$v = 0$ $w = 0$ (2D)

- Now consider x-momentum:

Not zero!

$g_x = g\sin(\theta)$

$$\rho\left(\underset{\text{Steady}}{\cancelto{0}{\frac{\partial u}{\partial t}}} + u\underset{\text{Fully Dev.}}{\cancelto{0}{\frac{\partial u}{\partial x}}} + \underset{v=0}{\cancel{v}}\frac{\partial u}{\partial y} + \underset{w=0}{\cancel{w}}\underset{\text{(2D)}}{\cancelto{0}{\frac{\partial u}{\partial z}}}\right) = -\cancelto{0}{\frac{\partial p}{\partial x}} + \mu\left(\underset{\text{Fully Dev.}}{\cancelto{0}{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \underset{\text{(2D)}}{\cancelto{0}{\frac{\partial^2 u}{\partial z^2}}}\right) + \rho g_x$$

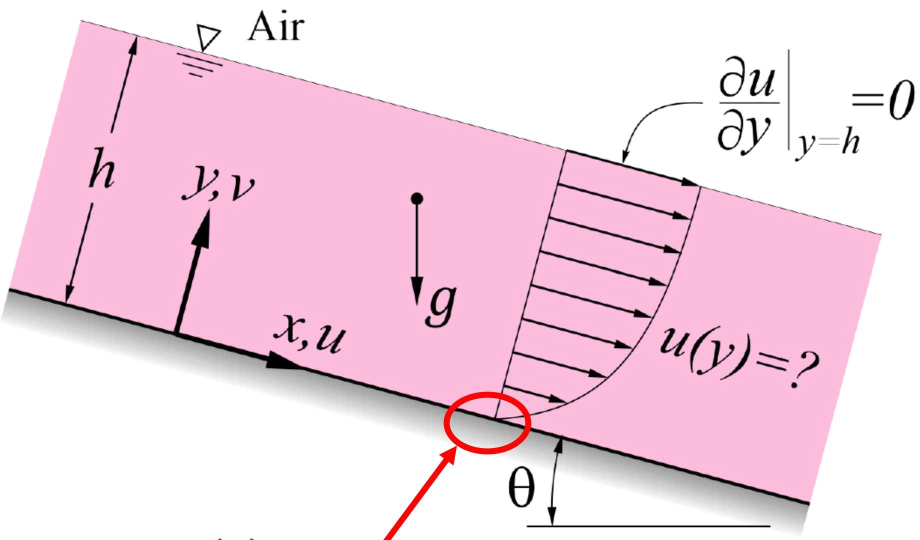


- Simplifying:

Full derivative: $u = u(y)\ only$

$$\frac{d^2u}{dy^2} = -\frac{\rho g sin(\theta)}{\mu} = constant$$

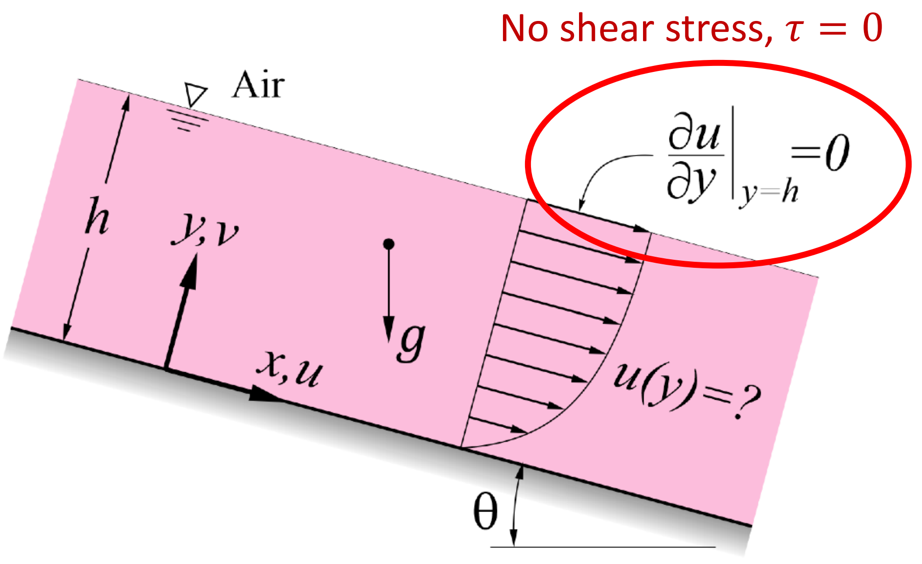
Navier-Stokes Equation Solution



No slip: $u(0) = 0$

- Result: $$\frac{d^2u}{dy^2} = -\frac{\rho g sin(\theta)}{\mu}$$
- Integrating with respect to y: $$\frac{du}{dy} = -\frac{\rho g sin(\theta)}{\mu}y + C_1$$
- Integrating again: $$u = -\frac{\rho g sin(\theta)}{2\mu}y^2 + C_1 y + \cancelto{0}{C_2} \quad (*)$$
- Constants C_1 & C_2 from velocity boundary conditions at $y = 0, h$
- No slip: Substitute $u = 0\ at\ y = 0$ into equation (*): $$\cancelto{0}{u} = -\frac{\rho g sin(\theta)}{2\mu}\cancelto{0}{y}^2 + C_1 \cancelto{0}{y} + C_2$$
- Thus: $$C_2 = 0$$

Navier-Stokes Equation Solution



No shear stress, $\tau = 0$

- Equation (*) becomes: $u = -\frac{\rho g sin(\theta)}{2\mu} y^2 + C_1 y$
- Use zero shear stress boundary condition at $y = h$ to get C_1
- After the 1st integration we had: $\frac{du}{dy} = -\frac{\rho g sin(\theta)}{\mu} y + C_1$
- Set $\frac{du}{dy} = 0$ at $y = h$: $\cancelto{0}{\frac{du}{dy}} = -\frac{\rho g sin(\theta)}{\mu} \cancelto{h}{y} + C_1$
- Thus: $0 = -\frac{\rho g sin(\theta)}{\mu} h + C_1$ → $C_1 = \frac{\rho g sin(\theta)}{\mu} h$

Navier-Stokes Equation Solution

- We now have: $$\boxed{C_1 = \frac{\rho gsin(\theta)}{\mu} h}$$

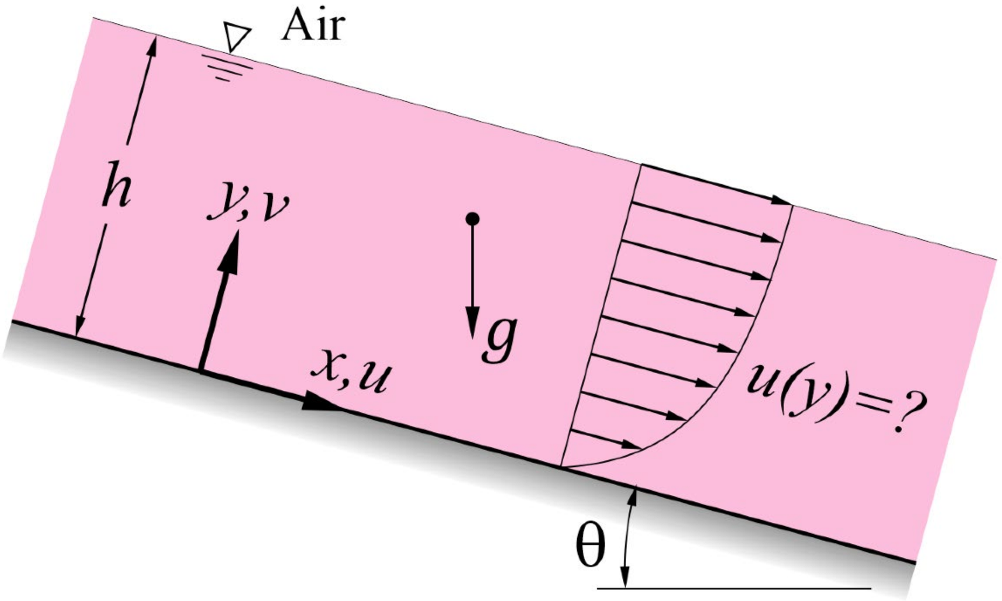
- Sub. into Eq. (*): $$u = -\frac{\rho gsin(\theta)}{2\mu} y^2 + C_1 y$$

- Result: $$u = -\frac{\rho gsin(\theta)}{2\mu} y^2 + \frac{\rho gsin(\theta)}{\mu} h\, y$$

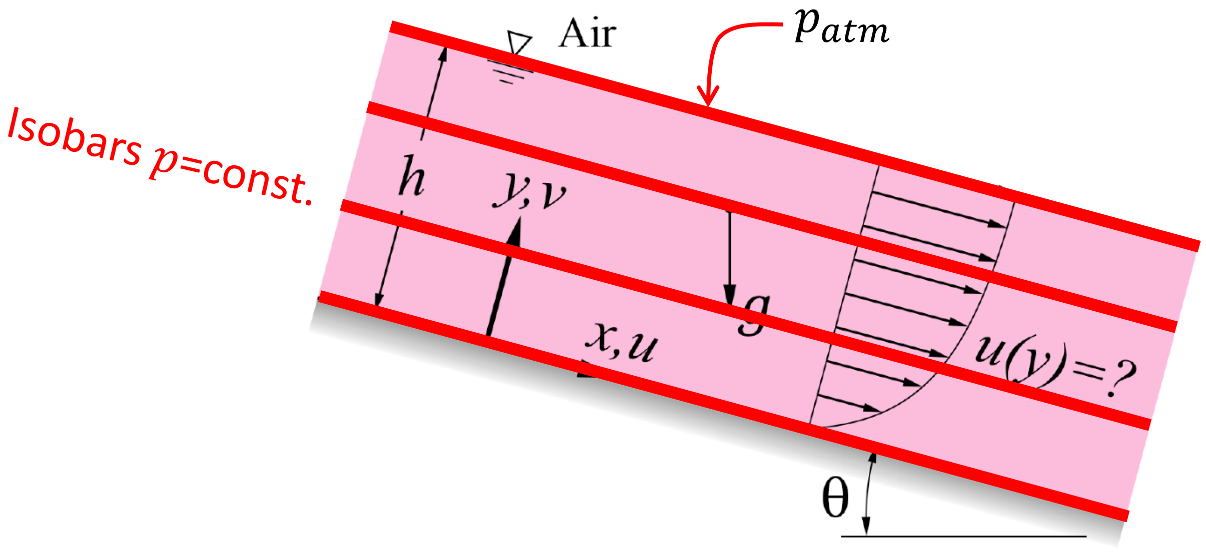
- Rearranging and collecting terms:

$$\boxed{u = \frac{\rho gsin(\theta)\, h^2}{\mu} \left(\frac{y}{h} - \frac{1}{2} \left(\frac{y}{h} \right)^2 \right)}$$

Answer (a)



Navier-Stokes Equation Solution



(b) The pressure gradient in the y-direction, $\dfrac{\partial p}{\partial y} = ?$

y-momentum equation:

$g_y = -g\cos(\theta)$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v}\cancel{\frac{\partial v}{\partial y}} + \cancel{w}\cancel{\frac{\partial v}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right) + \rho g_y$$

Steady
$v = 0$

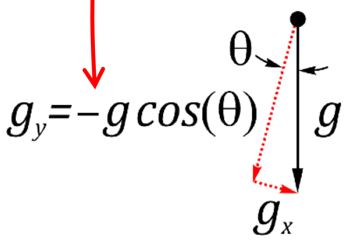
$v = 0$

$w = 0$

$v = 0$

$v = 0$

y-axis is upward



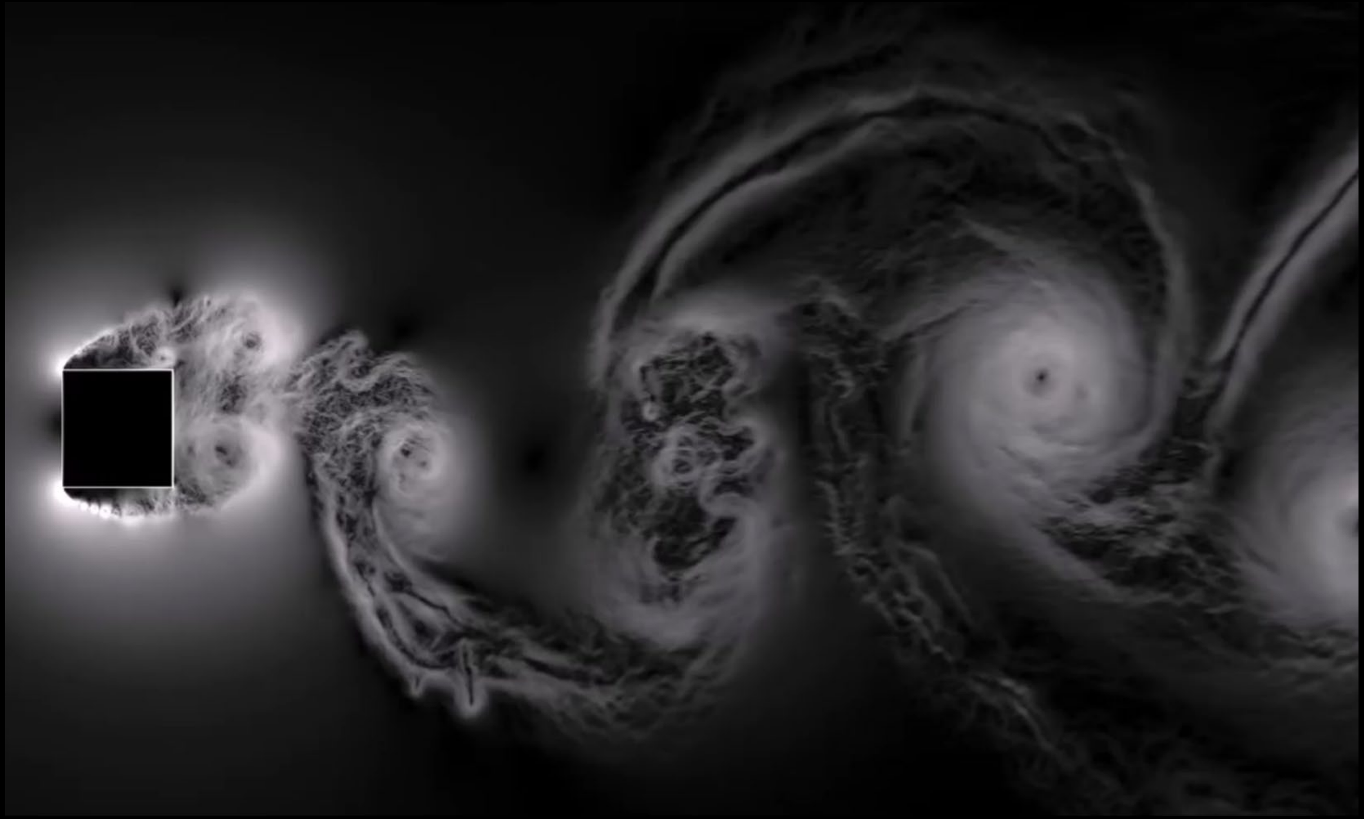
$$\boxed{\frac{dp}{dy} = -\rho g\cos(\theta)}$$

Answer (b)

Hydrostatic pressure gradient in y-direction

Direct Numerical Simulation (DNS) of the Navier-Stokes Equations. Turbulent flow over a square cylinder: Re=20,000, 324 million grid points. Visualization shows magnitude of the pressure gradient.

Credit: F. Xavier Trias
https://youtu.be/c8zKWaxohng



END NOTES

- All the videos (and pdf downloads) for this course available at: www.drdavidnaylor.net
- Presentation prepared and delivered by Professor David Naylor, 2023

Toronto Metropolitan University