MEC516/BME516
Fluid Mechanics I

Chapter 5
Recommended Problem Set

1. This wind tunnel test involves external isothermal flow. Dynamic similarity of the velocity fields over the vehicles requires matching of the Reynolds number:

$$Re_p = Re_m \qquad \frac{\rho_p V_p L_p}{\mu_p} = \frac{\rho_m V_m L_m}{\mu_m} \qquad V_p = \frac{\rho_m}{\rho_p}\frac{\mu_p}{\mu_m}\frac{L_m}{L_p}V_m$$

The fluid properties for air. Prototype (Denver):

$$\rho_p = \frac{p}{RT} = \frac{83{,}000\ Pa}{287\frac{J}{kgK}(-10+273K)} = 1.100\frac{kg}{m^3} \qquad \mu_p = 1.66x10^{-5}\frac{kg}{ms} \quad (interpolated)$$

Model (wind tunnel):

$$\rho_m = \frac{p}{RT} = \frac{101{,}300\ Pa}{287\frac{J}{kgK}(20+273K)} = 1.205\frac{kg}{m^3} \qquad \mu_m = 1.80x10^{-5}\frac{kg}{ms}$$

The wind speed for the model must be converted to meters per second:

$$V_m = 160\frac{mile}{h}\left(1609.3\frac{m}{mile}\right)\frac{h}{3600s} = 71.5\frac{m}{s}$$

Making the substitutions:

$$V_p = \left(\frac{1.205}{1.100}\right)\left(\frac{1.66}{1.80}\right)\left(\frac{1}{5}\right)71.5\frac{m}{s} = 14.4\frac{m}{s} \qquad (32.2\frac{miles}{h})$$

This is a relatively low speed. This calculation illustrates a technical limitation of wind tunnel testing with small models. Wind tunnel results are often correct for the mismatch of the Reynolds number -- an advanced topic, which is beyond the scope of this course.

2. See Figure 5.3 in the White textbook. For a smooth sphere, the dimensionless drag force (i.e., the drag coefficient, C_D) is a function of only the Reynolds number:

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A_f} = f(Re)$$

Thus, for dynamic similarity of the velocity field, the Reynolds numbers must be matched:

$$Re_p = Re_m \qquad \frac{\rho_p V_p L_p}{\mu_p} = \frac{\rho_m V_m L_m}{\mu_m} \qquad V_p = \frac{\rho_m}{\rho_p}\frac{\mu_p}{\mu_m}\frac{L_m}{L_p}V_m$$

Fluid properties for the model (in water): $\rho_m = 998\frac{kg}{m^3}, \quad \mu_m = 1.003x10^{-3}\frac{kg}{ms}$

Fluid properties for the prototype (in air):

$$\rho_p = \frac{p}{RT} = \frac{101{,}3000\ Pa}{287\frac{J}{kgK}(20 + 273K)} = 1.205\frac{kg}{m^3} \qquad \mu_p = 1.80x10^{-5}\frac{kg}{ms}$$

Making the substitutions, the wind speed for the balloon is:

$$V_p = \left(\frac{998}{1.205}\right)\left(\frac{1.80\ x10^{-5}}{1.003x10^{-3}}\right)\left(\frac{0.08}{1.5}\right)2.0\frac{m}{s} = 1.58\frac{m}{s}$$

At this wind speed, the flow field over the spheres will be dynamically similar. Hence, they will have the same drag coefficient. Thus, the prediction equation is:

$$\frac{F_{D,p}}{\frac{1}{2}\rho_p V_p^2 A_{f,p}} = \frac{F_{D,m}}{\frac{1}{2}\rho_m V_m^2 A_{f,m}}$$

Note that the areas in this equation are the projected frontal areas of the spheres (circles):

$$A_{f,p} = \frac{\pi\ D_p^2}{4}, \qquad A_{f,m} = \frac{\pi\ D_m^2}{4}$$

Rearranging the prediction equation:

$$F_{D,p} = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{D_p}{D_m}\right)^2 F_{D,m}$$

The drag force on the prototype is:

$$F_{D,p} = \left(\frac{1.205}{998}\right)\left(\frac{1.58}{2.0}\right)^2\left(\frac{1.5}{0.08}\right)^2 5.0\ N = 1.32\ N$$

3. Including the dependent variable, there are n=6 parameters with j=3 dimensions.

$$\{\tau_w\} = \left\{\frac{M}{T^2L}\right\} \quad \{U\} = \left\{\frac{L}{T}\right\} \quad \{u'\} = \left\{\frac{L}{T}\right\} \quad \{\rho\} = \left\{\frac{M}{L^3}\right\} \quad \{\delta\} = \{L\} \quad \left\{\frac{dp}{dx}\right\} = \left\{\frac{M}{T^2L^2}\right\}$$

Thus, this problem can be expressed using k=n-j=3 dimensionless parameters.

The problem statement indicates the j=3 repeating variables: ρ, U, δ. (It is required that these three variables cannot form a Pi group. This is true.)

$$\Pi_1 = \tau_w \, \rho^a U^b \delta^c = \left\{\frac{M}{T^2 L}\right\}\left\{\frac{M}{L^3}\right\}^a \left\{\frac{L}{T}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = -1, \qquad b = -2, \qquad -1 - 3a + b + c = 0 \quad c = 0$$

$$\Pi_1 = \frac{\tau_w}{\rho U^2}$$

$$\Pi_2 = u' \, \rho^a U^b \delta^c = \left\{\frac{L}{T}\right\}\left\{\frac{M}{L^3}\right\}^a \left\{\frac{L}{T}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = 0, \qquad b = -1, \qquad 1 - 3a + b + c = 0 \quad c = 0$$

$$\Pi_2 = \frac{u'}{U}$$

$$\Pi_3 = \frac{dp}{dx} \, \rho^a U^b \delta^c = \left\{\frac{M}{T^2 L^2}\right\}\left\{\frac{M}{L^3}\right\}^a \left\{\frac{L}{T}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = -1, \qquad b = -2, \qquad -2 - 3a + b + c = 0 \quad c = 1$$

$$\Pi_3 = \frac{\frac{dp}{dx}\delta}{\rho U^2}$$

This problem is described in dimensionless form as:

$$\Pi_1 = f(\Pi_2, \Pi_3) \qquad \frac{\tau_w}{\rho U^2} = f\left(\frac{u'}{U}, \frac{\frac{dp}{dx}\delta}{\rho U^2}\right)$$

4. Including the dependent variable, there are n=6 parameters with j=3 dimensions {L}, {M}, {T}. Note than an angle is a dimensionless quantity.

$$\{F\} = \left\{\frac{ML}{T^2}\right\} \quad \{\Omega\} = \left\{\frac{1}{T}\right\} \quad \{V\} = \left\{\frac{L}{T}\right\} \quad \{\rho\} = \left\{\frac{M}{L^3}\right\} \quad \{\mu\} = \{\frac{M}{LT}\}$$

You are told in the problem statement to use the j=3 repeating variables: ρ, V, D. (These three variables cannot form a Pi group).

$$\Pi_1 = F\,\rho^a V^b D^c = \left\{\frac{ML}{T^2}\right\}\left\{\frac{M}{L^3}\right\}^a\left\{\frac{L}{T}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = -1, \qquad b = -2, \qquad 1 - 3a + b + c = 0 \quad c = -2$$

$$\Pi_1 = \frac{F}{\rho V^2 D^2}$$

$$\Pi_2 = \Omega\,\rho^a V^b D^c = \left\{\frac{1}{T}\right\}\left\{\frac{M}{L^3}\right\}^a\left\{\frac{L}{T}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = 0, \qquad b = -1, \qquad -3a + b + c = 0 \quad c = 1$$

$$\Pi_1 = \frac{\Omega D}{V}$$

$$\Pi_3 = \mu\,\rho^a V^b D^c = \left\{\frac{M}{LT}\right\}\left\{\frac{M}{L^3}\right\}^a\left\{\frac{L}{T}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = -1, \qquad b = -1, \quad -1 - 3a + b + c = 0 \quad c = -1$$

$$\Pi_1 = \frac{\rho}{\rho V D} \quad \left(i.e.\ \frac{1}{Re}\right)$$

This problem be described in dimensionless form as:

$$\Pi_1 = f(\Pi_2, \Pi_3) \qquad \frac{F}{\rho V^2 D^2} = f(\frac{\Omega D}{V}, \frac{\rho}{\rho V D})$$

The Reynolds number is a well-known parameter in fluid mechanics. Also, the functional form in the above equation is unknown. So, any dimensionless parameter can be inverted without loss of generality. Thus, this dimensionless relationship would normally be written in an equivalent form as:

$$\frac{F}{\rho V^2 D^2} = f(\frac{\Omega D}{V}, \frac{\rho V D}{\mu})$$

5. (a) Including the dependent variable, there are n=5 parameters with j=3 dimensions {L}, {M}, {T}.

$$\{F_D\} = \left\{\frac{ML}{T^2}\right\} \quad \{V\} = \left\{\frac{L}{T}\right\} \quad \{\rho\} = \left\{\frac{M}{L^3}\right\} \quad \{\ell\} = \{L\} \quad \{\mu\} = \{\frac{M}{LT}\}$$

This problem can be described with two Pi parameters k=n-j=2. Thus, the three repeating variables will be V, ℓ and ρ.

$$\Pi_1 = F_D\, V^a\, \rho^b\, \ell^c = \left\{\frac{ML}{T^2}\right\}\left\{\frac{L}{T}\right\}^a \left\{\frac{M}{L^3}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = -2, \qquad b = -1, \qquad 1 + a - 3b + c = 0 \quad c = -2$$

$$\Pi_1 = \frac{F_D}{\rho V^2 \ell^2}$$

$$\Pi_2 = g\, V^a\, \rho^b\, \ell^c = \left\{\frac{L}{T^2}\right\}\left\{\frac{L}{T}\right\}^a \left\{\frac{M}{L^3}\right\}^b \{L\}^c = \{L\}^0\{M\}^0\{T\}^0$$

Solving for the exponents:

$$a = -2, \qquad b = 0, \qquad 1 + a - 3b + c = 0 \quad c = 1$$

$$\Pi_2 = \frac{\ell g}{V^2}$$

This is the inverse of the Froude number, as well-known dimensionless parameter in free surface (wave) flows.

This problem is described in dimensionless form as:

$$\Pi_1 = f(\Pi_2) \qquad \frac{F_D}{\rho V^2 \ell^2} = f(\frac{\ell g}{V^2})$$

The functional form in the above equation is unknown. So, any dimensionless parameter can be inverted without loss of generality. Thus, this dimensionless relationship would normally be written in an equivalent form in terms of the Froude number:

$$\frac{F_D}{\rho V^2 \ell^2} = f(\frac{V^2}{\ell g})$$

(b) Dynamic similarity for a free surface flow requires a matching of the Froude number. (Not the Reynolds number, as our analysis has just shown!)

$$Fr_p = Fr_m \qquad \frac{V_p^2}{\ell_p g} = \frac{V_m^2}{\ell_m g}$$

Both the model and prototype operate at the same value of g. So, the velocity of the prototype for dynamic similarity is:

$$V_p = \sqrt{\frac{\ell_p}{\ell_m}}\, V_m = \sqrt{\frac{50}{1}}\left(1.5\frac{m}{s}\right) = 10.61\frac{m}{s}$$

At this velocity, the Π_2 parameters are matched. So, the Π_1 parameters are also the same. This yields the prediction equation:

$$\frac{F_{D,p}}{\rho_p V_p^2 \ell_p^2} = \frac{F_{D,m}}{\rho_m V_m^2 \ell_m^2} \qquad F_{D,p} = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{\ell_p}{\ell_p}\right)^2 F_{D,m}$$

Noting that $\rho_p = \rho_m$, the wave drag on the full-scale (prototype) ship is:

$$F_{D,p} = (1.0)\left(\frac{10.61}{1.5}\right)^2\left(\frac{50}{1}\right)^2 14N = 1.75x10^6 N = 1750\ kN$$

6. We need to determine the drag coefficient on the cylinder using Figure 5.3 in the White textbook. The drag coefficient is a function of the Reynolds number.

Air properties at -40ºC and atmospheric pressure (Table A2): $\rho = 1.52\frac{kg}{m^3}$ $\quad \mu = 1.51x10^{-5}\frac{kg}{ms}$

$$Re = \frac{\rho\, DV}{\mu} = \frac{1.52\frac{kg}{m^3}\,(0.055\, m)35\frac{m}{s}}{1.51\ \mathrm{x}10^{-5}\ \frac{kg}{ms}} = 1.94\ x10^5$$

Using Figure 5.3, at this Reynolds number the drag coefficient is about: $C_D \approx 1.2$

The drag force on the cylinder is:

$$F_D = C_D\left(\frac{1}{2}\rho V^2 A_{frontal}\right)$$

Where the frontal area (per unit length) is: $A_{frontal} = LD = 1m(0.055m) = 0.055\ m^2$

So, the aerodynamic drag force on the cylinder (per unit length) is approximately:

$$F_D = 1.2\left(\frac{1}{2}(1.52\frac{km}{m^3})\ (35\frac{m}{s})^2\ (0.055m^2)\right) = 61\ N$$