MEC516/BME516: Fluid Mechanics I

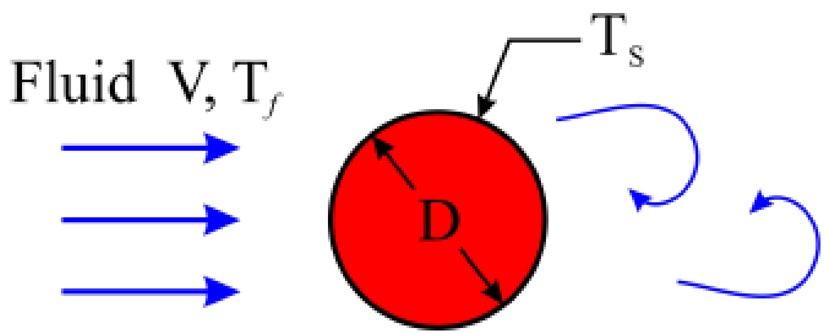
Chapter 5: Dimensional Analysis & Similarity
Part 4

RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering

Overview

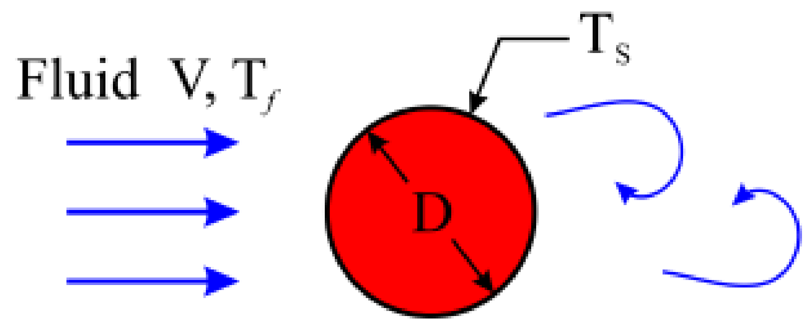
- Solved Example Dimensional Analysis
  - Pi parameters for forced convective heat transfer from an isothermal cylinder in cross flow.
  - Correlation of experimental data

- Dimensionless Parameters in Heat Transfer
  - Nusselt number
  - Reynolds number
  - Prandtl number



Wilhelm Nusselt (1882-1952)

Dimensional Analysis Example



Convective Heat Transfer from a Cylinder

The convective heat transfer $\dot{Q}$ from an isothermal cylinder can be expressed as:

$$\dot{Q} = h\ A\ \ (T_s - T_f\ ) \qquad \left(W, \frac{J}{s}\right)$$

where h is the convective heat transfer coefficient $(\frac{W}{m^2 K})$.

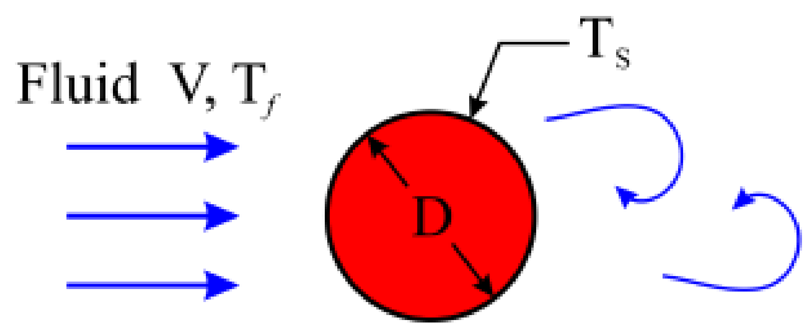
A is the surface area of the cylinder (m^2)

$(T_s - T_f)$ is the surface to fluid temperature difference $(K\ or\ \ ^oC)$

This is called *Newton's Law of Cooling*.

(The mechanical engineers will see this again in MEC701 Heat Transfer)

Example: Heat Transfer from a Cylinder



Problem Statement:

Newton's Law of Cooling $\dot{Q} = h\,A\,(T_s - T_f)$

The heat transfer coefficient h is a function of:

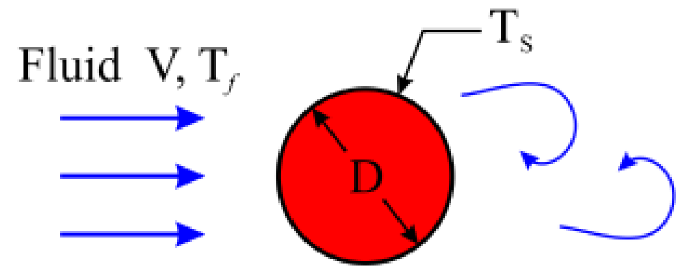
- Geometry: D
- Fluid Properties: specific heat, c_p dyn. viscosity, μ thermal conductivity, k density, ρ
- External Effects: free stream velocity, V

Given that $h = f_1(D, c_p, \mu, k, \rho, V)$ find the Pi terms that characterize forced convective heat transfer.

To obtain the classical dimensionless paramters use D, μ, k, ρ for the repeating variables. (Recall, the Pi parameters are not unique and may depend upon your selection of repeaters).

Example: Heat Transfer from a Cylinder

$h = f_1(D, c_p, \mu, k, \rho, V)$



Solution by Method of Repeating Variables (Six Steps)

Step 1: List the n variables in the problem

- You are told in the problem statement: $h = f_1(D, c_p, \mu, k, \rho, V)$
- So, we have $n = 7$ variables (including h)

✓All variables are independent.

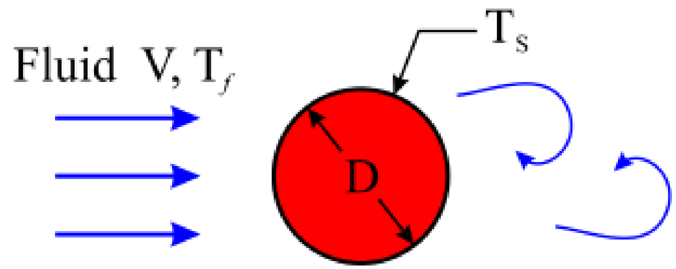
✓ Include relevant geometric effects, fluid properties, and external effects.
(gravity g is not included. Buoyancy effects are assumed to be small).

Step 2: Express the variables in terms of basic dimensions:

Start with the obvious ones: $\{D\} = \{L\}$ $\{V\} = \left\{\frac{L}{T}\right\}$ $\{\mu\} = \left\{\frac{M}{LT}\right\}$ $\{\rho\} = \left\{\frac{M}{L^3}\right\}$

$$h = f_1(D, c_p, \mu, k, \rho, V)$$

Example: Heat Transfer from a Cylinder



$$h \quad \frac{W}{m^2 K} = \frac{J}{s}\frac{1}{m^2 K} = \frac{N\,m}{s\,m^2 K} = \frac{kg\,m}{s^2}\frac{1}{s\,m\,K} \quad \left\{\frac{M}{T^3\Theta}\right\}$$

$$k \quad \frac{W}{m K} = \frac{J}{s}\frac{1}{m K} = \frac{N\,m}{s\,m\,K} = \frac{kg\,m}{s^2}\frac{1}{s K} \quad \left\{\frac{ML}{T^3\Theta}\right\}$$

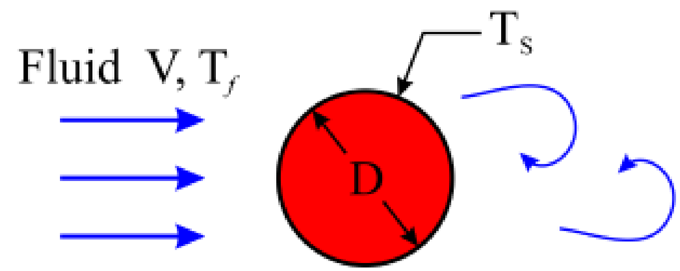
$$c_p \quad \frac{J}{kg\,K} = \frac{N\,m}{kg\,K} = \frac{kg\,m}{s^2}\frac{m}{kg\,K} \quad \left\{\frac{L^2}{T^2\Theta}\right\}$$

Thus: $\{D\} = \{L\}$, $\{V\} = \left\{\frac{L}{T}\right\}$, $\{\mu\} = \left\{\frac{M}{LT}\right\}$, $\{\rho\} = \left\{\frac{M}{L^3}\right\}$

$$\{h\} = \left\{\frac{M}{T^3\Theta}\right\}, \quad \{k\} = \left\{\frac{ML}{T^3\Theta}\right\}, \quad \{c_p\} = \left\{\frac{L^2}{T^2\Theta}\right\}$$

Example: Heat Transfer from a Cylinder

$h = f_1(D, c_p, \mu, k, \rho, V)$.)



$\{D\} = \{L\}, \quad \{V\} = \left\{\frac{L}{T}\right\}, \quad \{\mu\} = \left\{\frac{M}{LT}\right\}, \quad \{\rho\} = \left\{\frac{M}{L^3}\right\}$

$\{h\} = \left\{\frac{M}{T^3\Theta}\right\}, \quad \{k\} = \left\{\frac{ML}{T^3\Theta}\right\}, \quad \{c_p\} = \left\{\frac{L^2}{T^2\Theta}\right\}$

<u>Step 3</u>: Determine the number of Π parameters, $k = n - j$

We have $j = 4$ basic dimensions: M, L, T, Θ.

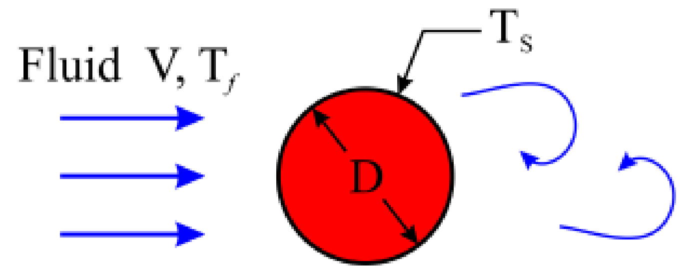
We have $n = 7$ variables.

∴ From Buckingham Pi Theorem: $k = n - j = 7 - 4 = 3$

We will get three dimensionless Pi terms.

Example: Heat Transfer from a Cylinder

$$h = f_1(D, c_p, \mu, k, \rho, V)$$



$$\{D\} = \{L\},\quad \{V\} = \left\{\frac{L}{T}\right\},\quad \{\mu\} = \left\{\frac{M}{LT}\right\},\quad \{\rho\} = \left\{\frac{M}{L^3}\right\}$$

$$\{h\} = \left\{\frac{M}{T^3\Theta}\right\},\quad \{k\} = \left\{\frac{ML}{T^3\Theta}\right\},\quad \{c_p\} = \left\{\frac{L^2}{T^2\Theta}\right\}$$

Step 4: Select "$j = 4$" repeating variables from the "$n = 7$" variables

To get the classical Pi parameters, you are told to pick D, μ, k, ρ as the repeating variables.

Checks:

✓ All reference dimensions M, L, T, Θ must be included in the "repeaters".

✓ The repeating variables cannot themselves form a dimensionless product. A Rigorous check:

$$D^a \mu^b k^c \rho^d = \{L\}^a \left\{\frac{M}{LT}\right\}^b \left\{\frac{ML}{T^3\Theta}\right\}^c \left\{\frac{M}{L^3}\right\}^d = M^0 L^0 T^0 \Theta^0$$

Θ: $\therefore c = 0$

T: $-b - 3c = 0 \;\therefore b = 0$

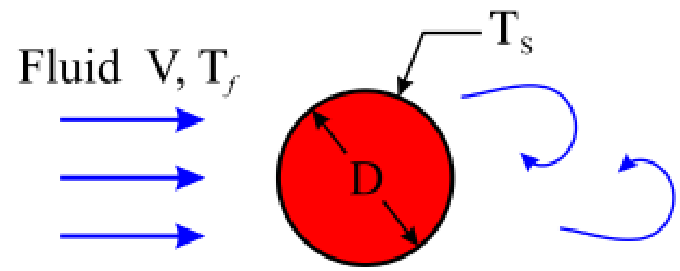
M: $b + c + d = 0 \;\therefore d = 0$

L: $a - b + c - 3d = 0 \;\therefore a = 0$

There are no non-zero values of a, b, c and d that can form a dimensionless product.

Example: Heat Transfer from a Cylinder

$h = f_1(D, c_p, \mu, k, \rho, V)$



Step 5: Form $k = n - j = 7 - 4 = 3$ Pi terms.

- We have three non-repeating variables: h, V, c_p

start with the dependent variable to form Π_1

$$\Pi_1 = hD^a\mu^b k^c \rho^d = \left\{\frac{M}{T^3\Theta}\right\}\{L\}^a\left\{\frac{M}{LT}\right\}^b\left\{\frac{ML}{T^3\Theta}\right\}^c\left\{\frac{M}{L^3}\right\}^d = M^0L^0T^0\Theta^0$$

4 repeating variables

Exponents for Θ: $-1 - c = 0 \quad \therefore c = -1$

Exponents for T: $-3 - b - 3c = 0 \quad b = -3 - 3c = -3 - 3(-1) \quad \therefore b = 0$

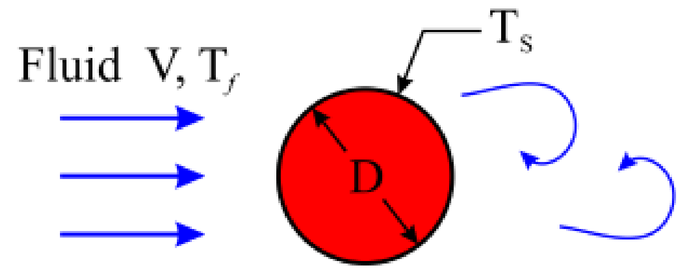
Exponents for M: $1 + b + c + d = 0 \quad d = -1 - b - c = -1 - 0 - (-1) \quad \therefore d = 0$

Exponents for L: $a - b + c - 3d = 0 \quad a = b - c + 3d \quad a = 0 - (-1) + 3(0) \therefore a = 1$

$\Pi_1 = hD^1\mu^0k^{-1}\rho^0$ $\qquad \Pi_1 = \frac{hD}{k}$

$h = f_1(D, c_p, \mu, k, \rho, V)$

Example: Heat Transfer from a Cylinder



Step 5: Form $k = n - j = 7 - 4 = 3$ Pi terms.

- We have three non-repeating variables: h, V, c_p

$$\Pi_2 = VD^a\mu^b k^c \rho^d = \left\{\frac{L}{T}\right\}\{L\}^a\left\{\frac{M}{LT}\right\}^b\left\{\frac{ML}{T^3\Theta}\right\}^c\left\{\frac{M}{L^3}\right\}^d = M^0L^0T^0\Theta^0$$

Exponents for Θ: $\therefore c = 0$

Exponents for T: $-1 - b - 3c = 0 \quad b = -1 - 3c \quad \therefore b = -1$

Exponents for M: $\mathrm{b} + c + d = 0 \quad d = -b - c \quad \therefore \quad d = 1$

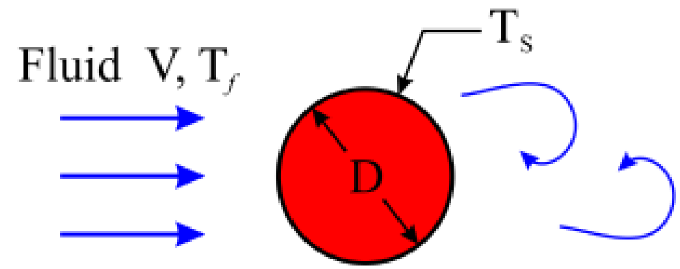
Exponents for L: $1 + a - b + c - 3d = 0 \quad a = -1 + b - c + 3d \quad \therefore a = -1 - 1 + 3 = 1$

$$\Pi_2 = VD^1\mu^{-1}k^0\rho^1$$

$$\Pi_2 = \frac{VD\rho}{\mu} = Re$$

(Recognize as Reynolds number)

$h = f_1(D, c_p, \mu, k, \rho, V)$



Example: Heat Transfer from a Cylinder

Step 5: Form $k = n - j = 7 - 4 = 3$ Pi terms.

- We have three non-repeating variables: h, V, c_p

$$\Pi_3 = c_p D^a \mu^b k^c \rho^d = \left\{\frac{L^2}{T^2\Theta}\right\}\{L\}^a \left\{\frac{M}{LT}\right\}^b \left\{\frac{ML}{T^3\Theta}\right\}^c \left\{\frac{M}{L^3}\right\}^d = M^0 L^0 T^0 \Theta^0$$

Exponents for Θ: $-1 - \mathrm{c} = 0 \therefore c = -1$

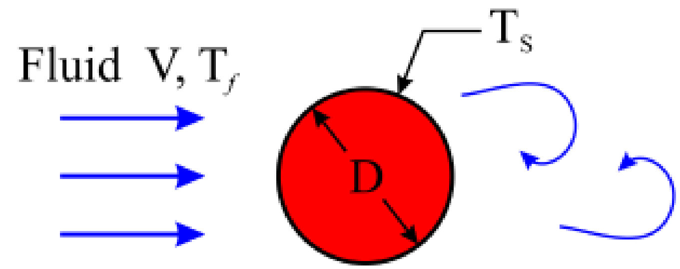
Exponents for T: $-2 - b - 3c = 0 \quad b = -2 - 3c = -2 - 3(-1) \quad \therefore b = 1$

Exponents for M: $\mathrm{b} + c + d = 0 \quad d = -b - c = -1 - (-1) \quad \therefore \; d = 0$

Exponents for L: $2 + a - b + c - 3d = 0 \quad a = -2 + b - c + 3d \quad \therefore a = -2 + 1 - (-1) = 0$

$$\Pi_3 = c_p D^0 \mu^1 k^{-1} \rho^0 \qquad \Pi_3 = \frac{c_p \mu}{k}$$

$$h = f_1(D, c_p, \mu, k, \rho, V)$$



Example: Heat Transfer from a Cylinder

We have shown: $\Pi_1 = \frac{hD}{k}$ $\Pi_2 = \frac{VD\rho}{\mu}$ $\Pi_3 = \frac{c_p\mu}{k}$

Step 6: Express the final form: $\Pi_1 = f_2(\Pi_2, \Pi_3, \ldots.)$

- Put the dependent variable (h) in the numerator of Π_1.
- So, we get the result:

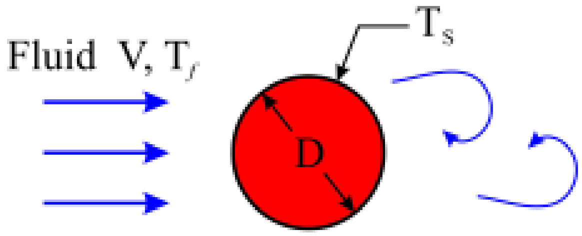
$$\frac{hD}{k} = f_2\left(\frac{VD\rho}{\mu}, \frac{c_p\mu}{k}\right)$$

Ans.

- That is as far as you would go on an exam. But, I will make some extra comments here.

Example: Heat Transfer from a Cylinder

$$h = f_1(D, c_p, \mu, k, \rho, V)$$

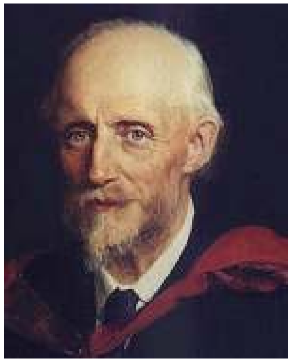


$$\frac{hD}{k} = f_2\left(\frac{VD\rho}{\mu}, \frac{c_p\mu}{k}\right)$$ Ans.

Π_1 is the Nusselt number: $Nu = \frac{hD}{k}$

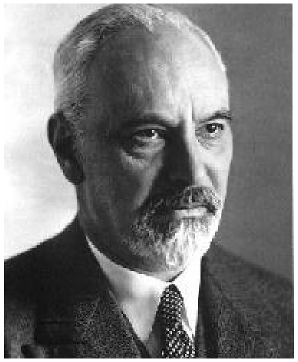
Wilhelm Nusselt (1882-1952)

Π_2 is the Reynolds number: $Re = \frac{VD\rho}{\mu}$



Osborne Reynolds (1842-1912)

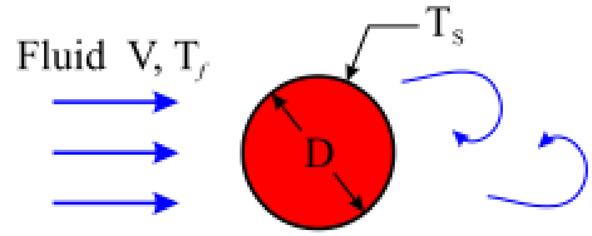
Π_3 is the Prandtl number: $Pr = \frac{c_p\mu}{k}$



Ludwig Prandtl (1875-1953)

Example: Heat Transfer from a Cylinder

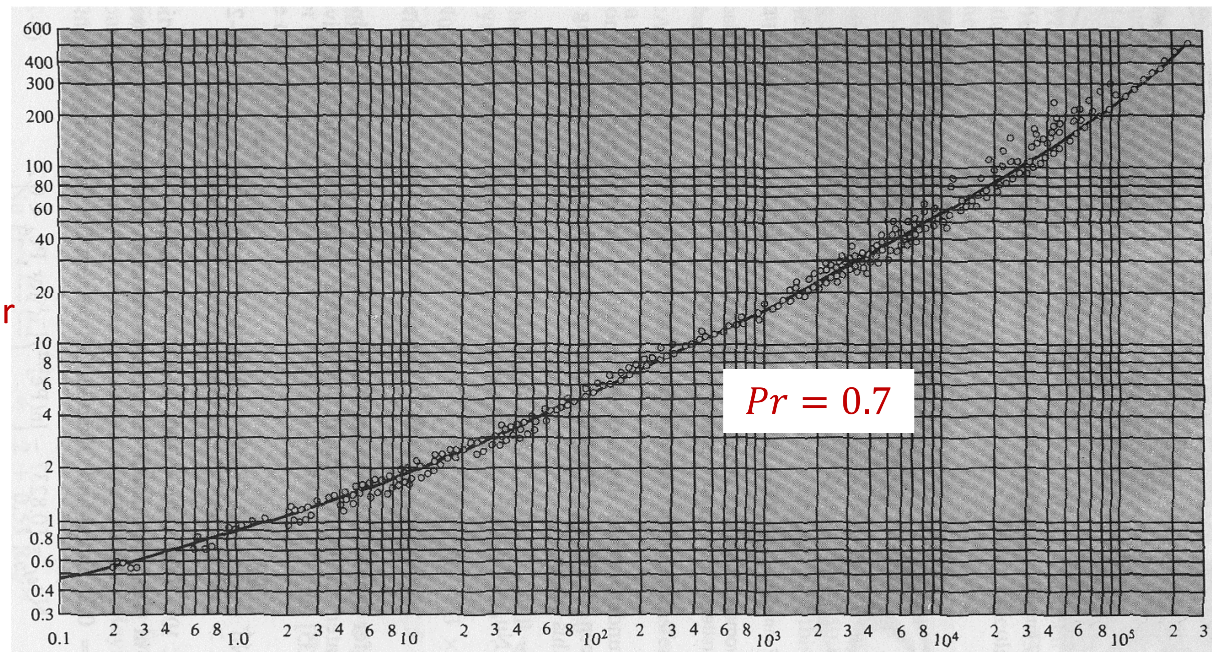
$$h = f_1(D, c_p, \mu, k, \rho, V)$$



$$\frac{hD}{k} = f_2\left(\frac{VD\rho}{\mu}, \frac{c_p\mu}{k}\right) \qquad Nu = f_2(Re, Pr)$$

For air (fixed Pr=0.7) this complex problem can be reduced to a single curve!

Nusselt number $Nu = \frac{hD}{k}$



Reynolds number $Re = \frac{VD\rho}{\mu}$

http://www.herbrich.com

END NOTES

Presentation prepared and delivered by Dr. David Naylor.