MEC516/BME516: Fluid Mechanics I

Chapter 5: Dimensional Analysis & Similarity
Part 3

RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering

Dimensional Analysis Example

<u>Solved Example: "Stokes" Flow over a Sphere</u>

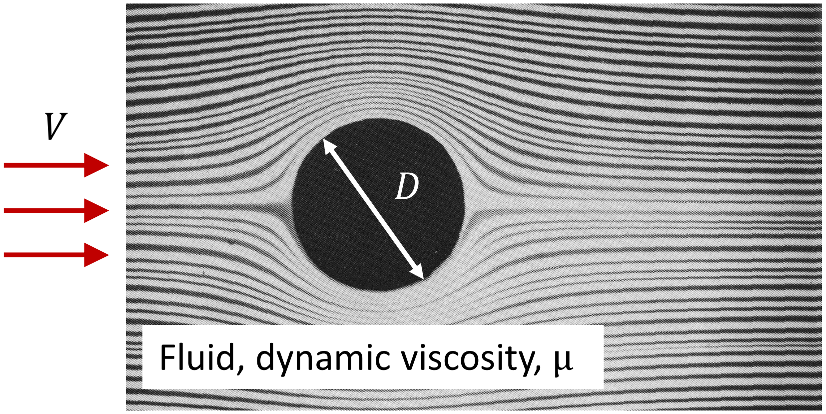
As you learned in Lab #1, at very low velocities the drag force F_D on a sphere is only a function of the sphere diameter D, the fluid dynamic viscosity μ, and the steady fluid velocity V. For so-called *creeping* flow:

$$F_D = f_1(D, V, \mu)$$

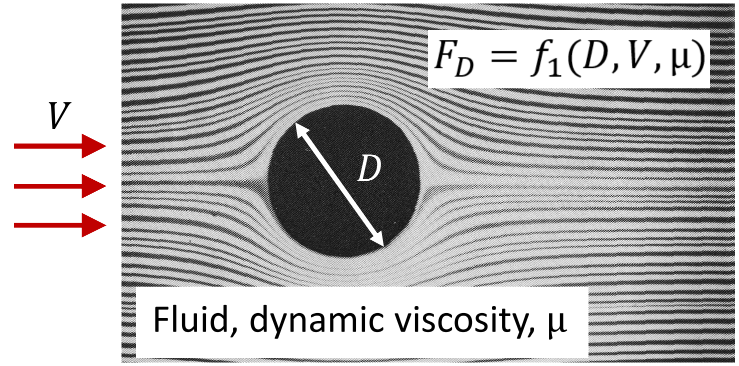
Use dimensional analysis to determine the form of the Stokes' equation.

(Recall the famous result is: $F_D = 3\pi\mu VD$)

G.G. Stokes (1819-1903)



Example: Stokes Flow over a Sphere



Method of Repeating Variables (Six Steps)

(such a formal approach is not necessary on a test)

Step 1: List the n variables in the problem

- From the problem statement: $F_D = f_1(D, V, \mu)$
- So, we have $n = 4$ variables (including F_D)

✓ All variables are independent.

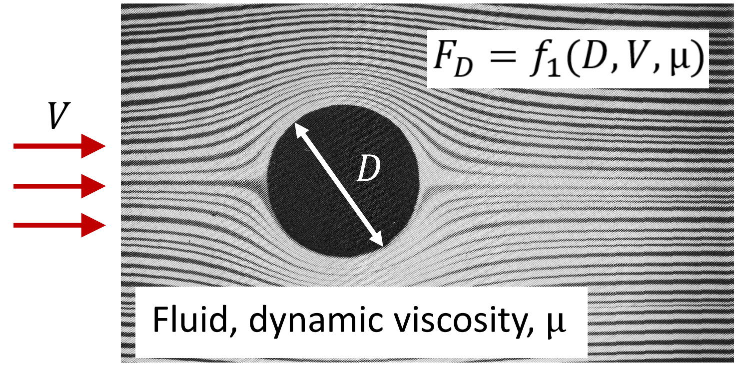
✓ Include relevant geometric effects D , fluid properties μ, and external effects V.

(Aside: ρ is not included because inertial effects are neglected for creeping flow. Fluid acceleration ≈ 0)

Step 2: Express the variables in terms of basic dimensions:

$$\{F_D\} = \left\{\frac{ML}{T^2}\right\} \qquad \{D\} = \{L\} \qquad \{V\} = \left\{\frac{L}{T}\right\} \qquad \{\mu\} = \left\{\frac{M}{LT}\right\}$$

Example: Stokes Flow over a Sphere



Step 2: Express the variables in terms of basic dimensions:

$\{F_D\} = \left\{\frac{ML}{T^2}\right\}$ $\{D\} = \{L\}$ $\{V\} = \left\{\frac{L}{T}\right\}$ $\{\mu\} = \left\{\frac{M}{LT}\right\}$

Step 3: Determine the number of Π parameters, $k = n - j$

We have $j = 3$ basic dimensions: M, L, T.

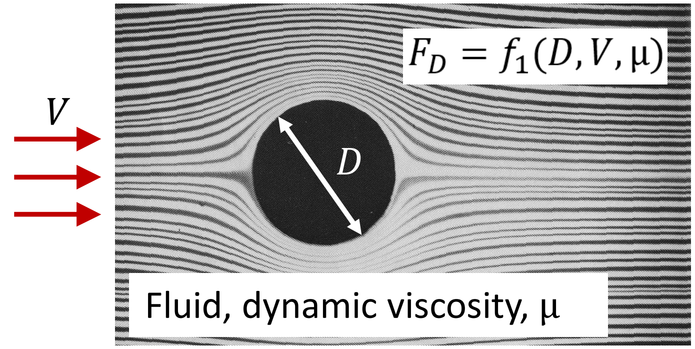
We have $n = 4$ variables.

∴ From Buckingham Pi Theorem: $k = n - j = 4 - 3 = 1$

We get only one dimensionless Pi term! (An unusual case. That's why I selected this problem.)

Example: Stokes Flow over a Sphere

$\{F_D\} = \left\{\frac{ML}{T^2}\right\}$ $\{D\} = \{L\}$ $\{V\} = \left\{\frac{L}{T}\right\}$ $\{\mu\} = \left\{\frac{M}{LT}\right\}$



Step 4: Select "$j = 3$" repeating variables from the "$n = 4$" variables

- Recall: Should not pick the independent parameter (F_D) as a “repeater”.

- So, we have no choice. The repeaters are: $\{D\} = \{L\}$ $\{V\} = \left\{\frac{L}{T}\right\}$ $\{\mu\} = \left\{\frac{M}{LT}\right\}$

Checks:

✓All reference dimensions M, L, T must be included in the “repeaters”.

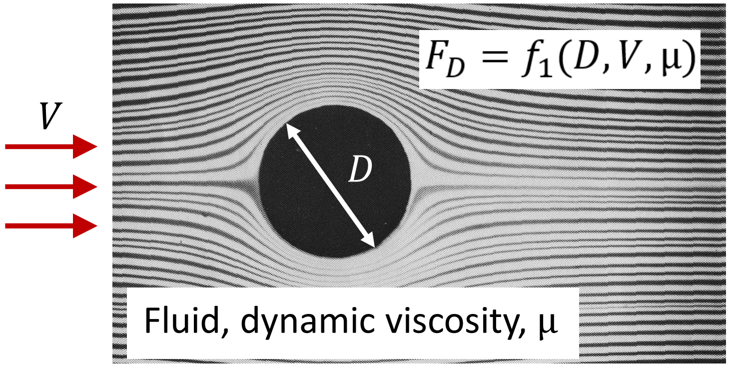
✓ The repeating variables cannot themselves form a dimensionless product (by inspection).

(A dimensionless parameter cannot include μ because of $\{M\}$. V and D cannot form a dimensionless parameter because of $\{T\}$.)

Example: Stokes Flow over a Sphere

$\{F_D\} = \left\{\frac{ML}{T^2}\right\}$ $\{D\} = \{L\}$ $\{V\} = \left\{\frac{L}{T}\right\}$ $\{\mu\} = \left\{\frac{M}{LT}\right\}$

3 repeating variables



Step 5: Form $k = n - j = 4 - 3 = 1$ Pi term, using the one non-repeating variable, F_D :

$$\Pi_1 = F_D D^a V^b \mu^c = \left(\frac{ML}{T^2}\right)(L)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{LT}\right)^c = L^0 M^0 T^0$$

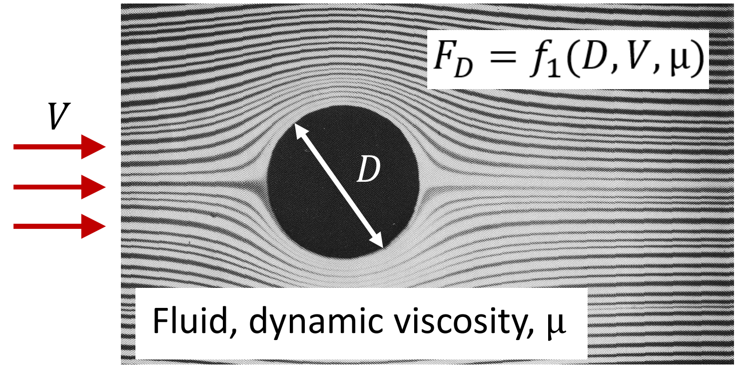
3 repeating variables

- Now we evaluate the exponents a, b, c that make Π_1 dimensionless.

Example: Stokes Flow over a Sphere

$$\Pi_1 = F_D D^a V^b \mu^c = \left(\frac{ML}{T^2}\right)(L)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{LT}\right)^c = L^0 M^0 T^0$$

match exponents, left = right



- Matching the exponents:

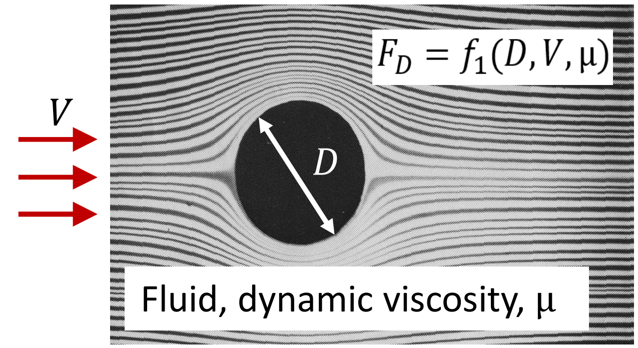
Exponents for M: $1 + c = 0$ $\quad \therefore c = -1$

Exponents for T: $-2 - b - c = 0$ $\quad b = -2 - c = -2 - (-1)$ $\quad \therefore b = -1$

Exponents for L: $1 + a + b - c = 0 \rightarrow a = c - b - 1$ $\quad \therefore a = -1 - (-1) - 1 = -1$

So, making the substitutions: $\Pi_1 = F_D D^{-1} V^{-1} \mu^{-1}$ $\qquad \Pi_1 = \frac{F_D}{D\,V\,\mu}$

Example: Stokes Flow over a Sphere



$$\Pi_1 = \frac{F_D}{D\,V\,\mu}$$

<u>Step 6</u>: Express the final form: $\Pi_1 = f_2(\Pi_2, \Pi_3, \ldots.)$

In this case there are no other Pi terms. So, it follows that the function f_2 is a constant, C.

So, we get the result: $\boxed{F_D = C\,\mu V D}$ Ans.

C is independent of the fluid, sphere diameter and fluid velocity for creeping flow (Re<<1).

We cannot obtain the constant C from dimensional analysis. But, technically, only a <u>single experiment</u> would be required to show that:

$$C = 3\pi \cong 9.42 \quad \text{(dimensionless)}$$

(Such experiments agree with Stokes' famous result that $C = 3\pi$, obtained by analytical solution of the Navier-Stokes and continuity equations for creeping flow, where convective acceleration terms are neglected).

END NOTES

Presentation prepared and delivered by Dr. David Naylor.