MEC516/BME516: Fluid Mechanics I

Chapter 5: Dimensional Analysis & Similarity
Part 2

RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering

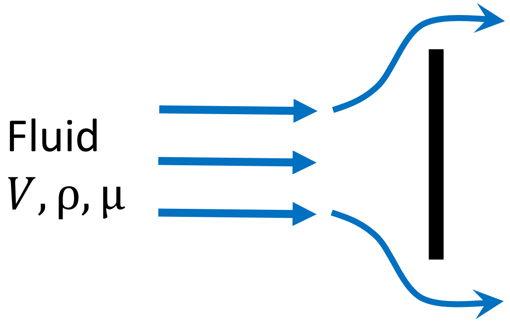
Overview

Dimensional Analysis

- Introduction
  - General utility of dimensional analysis
- Buckingham Pi Theorem
  - The method of repeating variables
- Example
  - Dimensionless parameters for the drag force ("form drag") for flow over a rectangular plate

http://www.blwtl.uwo.ca/

www.nasa.gov



Introduction to Dimensional Analysis

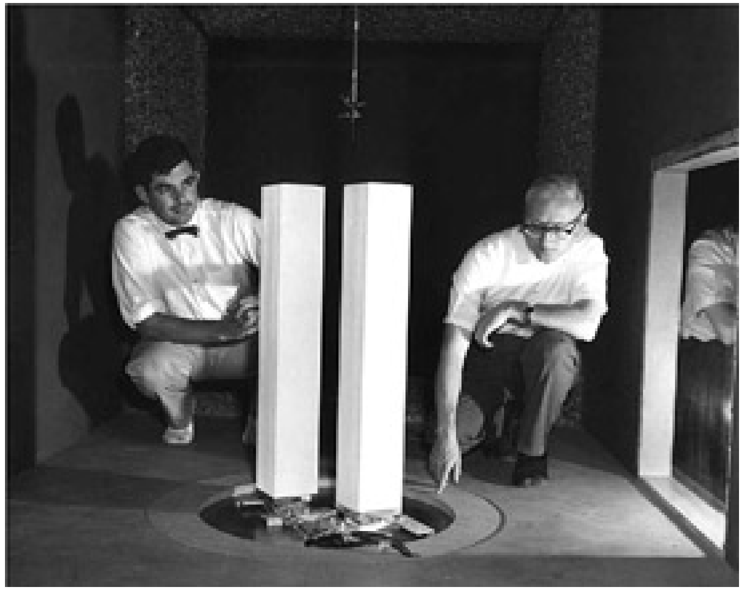
- Many practical fluid flow problems are too complex to solve analytically, e.g. turbulent flow
  - Must resort to experiments or approximate modelling (CFD)

- Question arises: How should we present the resulting data?

- This chapter will show that data are best presented in dimensionless form.
  - most compact form (least effort & expense)
  - most generality (not only for specific conditions)
  - similarity (or similitude) used to relate results on a small scale physical model to the full scale problem, e.g. wind tunnel testing "scale up". (Scaling will be discussed in an upcoming video).

- We will determine the dimensionless parameters using *Dimensional Analysis*

www.nasa.gov



Boundary Layer Wind Tunnel (UWO)

http://www.alumni.westernu.ca/

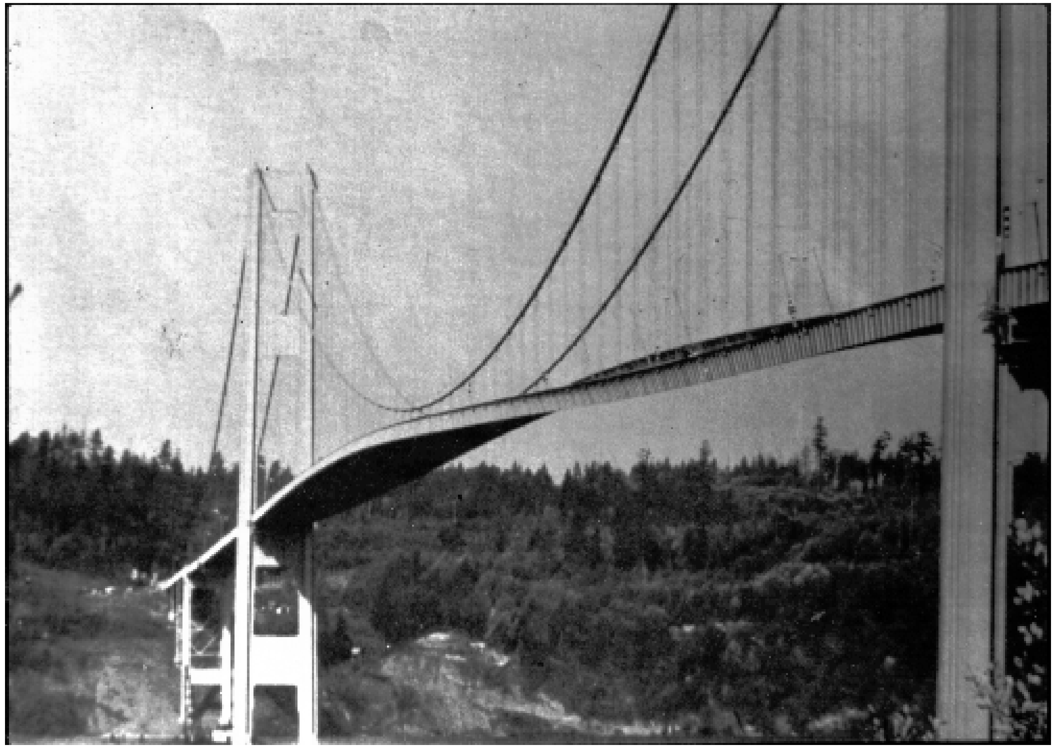
Boundary Layer Wind Tunnel (Univ. of Western Ontario)

- A pioneering centre for wind tunnel testing of large structures (buildings, bridges, towers, etc.)
- Similarity is used to scaled up data from small physical models to predict full scale behaviour.
- Confederation Bridge, New Brunswick to PEI (1997). Wind tunnel testing to predict:
  - Wind loads on span
  - Affect of wind on cars on the driving deck. For what conditions can you still use the bridge?
  - Flow-induced vibrations, fluid-structure interactions. Called “aeroelestic testing”

http://www.blwtl.uwo.ca/

Bridge at Tacoma Narrows (1940)

- Flow-induced vibration destroyed an early suspension bridge (Puget Sound, Oregon, USA)
- Nowadays large bridges are routinely wind tunnel tested using (small) scaled models



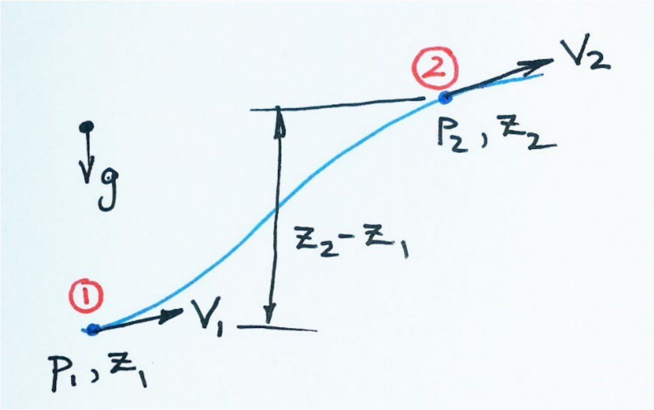
Principle of Dimensional Homogeneity (PDH)

Principle of Dimensional Homogeneity is the basis of dimensional analysis.

Statement of PDH:

A complete equation that expresses the relationship between variables in a physical process must be dimensionally homogeneous. Additive terms must have the same dimensions.

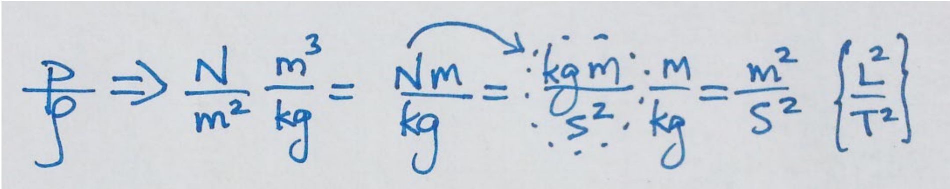
For example, recall the Bernoulli equation applied on a streamline:



$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + g z_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + g z_2$$

$$\left\{\frac{L^2}{T^2}\right\} \quad \left\{\frac{L^2}{T^2}\right\} \quad \left\{\frac{L^2}{T^2}\right\} \quad \left\{\frac{L^2}{T^2}\right\} \quad \left\{\frac{L^2}{T^2}\right\} \quad \left\{\frac{L^2}{T^2}\right\}$$

No need to memorize!



Buckingham Pi Theorem

Statement of the Buckingham Pi Theorem:

If an equation involving n variables is dimensionally homogeneous, it can be reduced to a function of $k = n - j$ dimensionless products, where j is the minimum number of reference dimensions needed to describe the variables.

Consider a problem with n variables:

$$u_1 = f_1(u_2, u_3 \ldots, u_n)$$

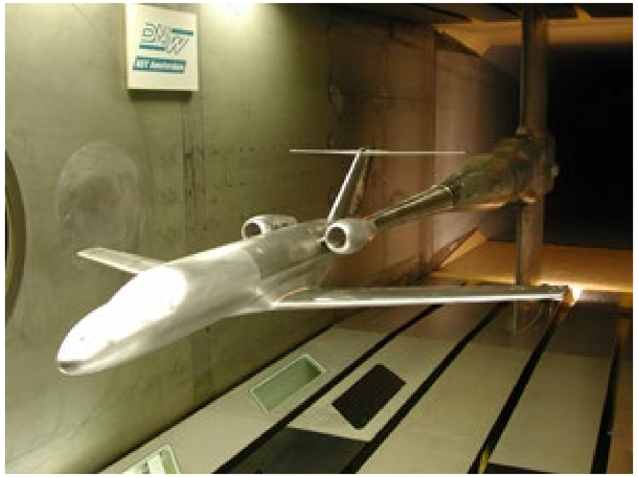
Dependent variable

PDH requires that the dimensions on the left and right side of the equation must be the same. Such an equation can be expressed as a set of dimensionless products:

$$\Pi_1 = f_2(\Pi_2, \Pi_3, \ldots \Pi_k)$$

- The problem can be expressed in terms of $k = (n - j)$ dimensionless terms. Get a reduction in number of variables. (The symbol Π is used because the dimensionless terms are products.)

Buckingham Pi Theorem



Method of Repeating Variables (Six Steps)

Step 1: List the n variables in the problem.

- Based on your knowledge of the physics of the problem.
- This is vitally important to get all quantities (otherwise the Π parameters will be wrong!)
- Variables must describe the applicable effects. Three general categories:
  - Geometry effects, e.g. pipe diameter, surface roughness, etc.
  - Fluid properties, e.g. viscosity, density, surface tension, etc.
  - External effects, e.g. driving pressure gradient.
- All variables must be independent. For example:
  - Do not include both pipe diameter D and cross sectional area A_c (since $A_c = \pi D^2/4$).
  - If fluid density and specific weight are important, list only two of: ρ, γ, g (since $\gamma = \rho g$)

Buckingham Pi Theorem

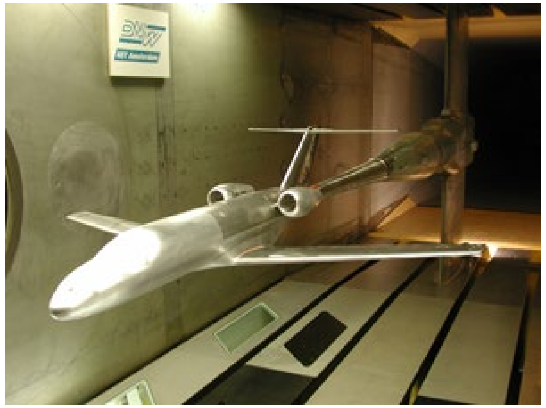
Step 2: Express each of the variables in terms of basic dimensions

- Use either (M, L, T, Θ) or (F, L, T, Θ). Recall the from **F**=m**a**: {F}={M L T^{-2}}
- See Table in Chapter 1:

Secondary dimension	SI unit	BG unit	Conversion factor
Area $\{L^2\}$	m^2	ft^2	$1\ m^2 = 10.764\ ft^2$
Volume $\{L^3\}$	m^3	ft^3	$1\ m^3 = 35.315\ ft^3$
Velocity $\{LT^{-1}\}$	m/s	ft/s	1 ft/s = 0.3048 m/s
Acceleration $\{LT^{-2}\}$	m/s^2	ft/s^2	$1\ ft/s^2 = 0.3048\ m/s^2$
Pressure or stress $\{ML^{-1}T^{-2}\}$	$Pa = N/m^2$	lbf/ft^2	$1\ lbf/ft^2 = 47.88\ Pa$
Angular velocity $\{T^{-1}\}$	s^{-1}	s^{-1}	$1\ s^{-1} = 1\ s^{-1}$
Energy, heat, work $\{ML^2T^{-2}\}$	$J = N \cdot m$	$ft \cdot lbf$	$1\ ft \cdot lbf = 1.3558\ J$
Power $\{ML^2T^{-3}\}$	W = J/s	$ft \cdot lbf/s$	$1\ ft \cdot lbf/s = 1.3558\ W$
Density $\{ML^{-3}\}$	kg/m^3	$slugs/ft^3$	$1\ slug/ft^3 = 515.4\ kg/m^3$
Viscosity $\{ML^{-1}T^{-1}\}$	$kg/(m \cdot s)$	$slugs/(ft \cdot s)$	$1\ slug/(ft \cdot s) = 47.88\ kg/(m \cdot s)$
Specific heat $\{L^2T^{-2}\Theta^{-1}\}$	$m^2/(s^2 \cdot K)$	$ft^2/(s^2 \cdot {}^\circ R)$	$1\ m^2/(s^2 \cdot K) = 5.980\ ft^2/(s^2 \cdot {}^\circ R)$

- No need to memorize! If you know the units of a variable, you can deduce its dimensions

Buckingham Pi Theorem



Step 3: Determine the number of Π parameters

- The Buckingham Pi Theorem says that the number of Pi terms is $n - j$
  - n is the number of independent variables
  - j is the number of basic dimensions in the variables
- j is found from inspection of the variable dimensions in Step 2.

An Aside: In rare cases the basic dimensions appear in combinations. So, the number of dimensions (j) can be less then the number of dimensions used in the variables. See textbook for details.

Step 4: Select "j" repeating variables from the "n" variables

- These "repeaters" will be in all the Pi terms.
- Rules for selecting the repeating variables:
  - All the basic dimensions of the problem must be included in these "repeaters".
  - The repeating variables cannot themselves form dimensionless product.
  - Do not pick the parameter of interest as a "repeater" (it will get buried in all the Pi terms).

Buckingham Pi Theorem

Step 5: Form $k = n - j$ Pi terms, with repeating variables raised to an arbitrary exponent

- Each of the k Pi terms will have the form:

$$\Pi_i = \underbrace{u_i}_{\text{"k" non-repeating variable}} \underbrace{u_1^{a1} u_2^{a2} u_3^{a3} \ldots u_j^{aj}}_{\text{"j" repeating variables}} \qquad i = 1 \ldots k$$

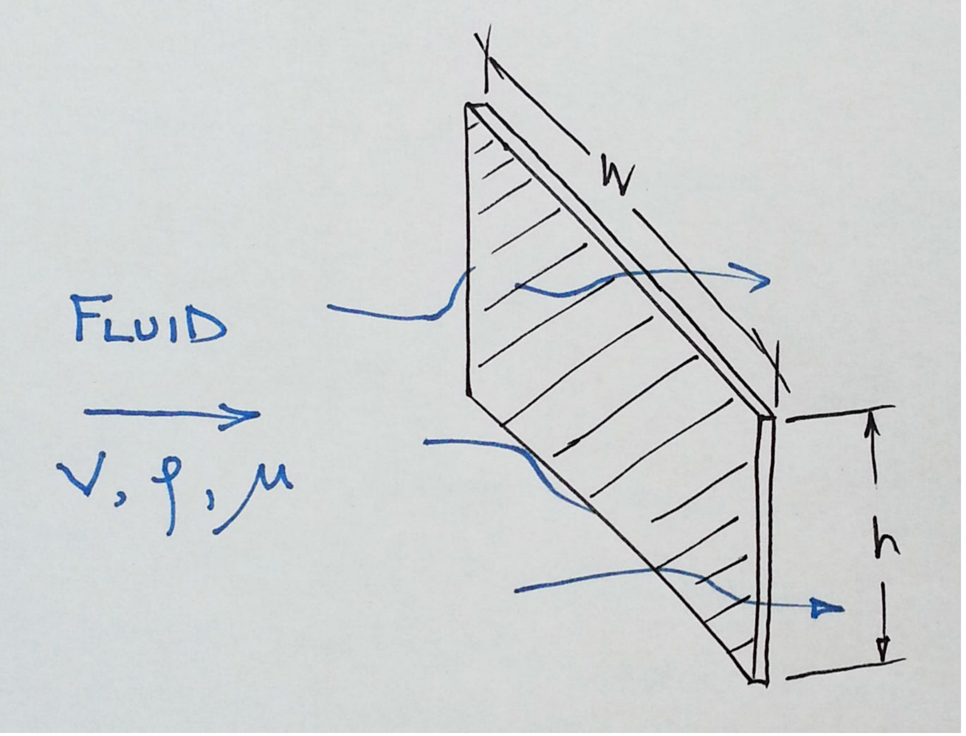
- The exponents $a1, a2, aj$ are determined so the Pi product is dimensionless.
- One-by-one, we then solve for the exponents in the Pi terms: $\Pi_1, \Pi_2, \ldots, \Pi_k$.

Step 6: Express the final form: $\Pi_1 = f_2(\Pi_2, \Pi_3, \ldots \Pi_k)$

- Put the dependent variable (the variable of interest) in the numerator of Π_1.
- The functional relationship can then be found by experiment (with greatly reduced effort).

Example: Aerodynamic Drag on a Plate

Consider that we want to experimentally characterize the drag force F_D on a rectangular plate produced by a flow of fluid perpendicular to the surface, e.g. wind loads on billboard signs.

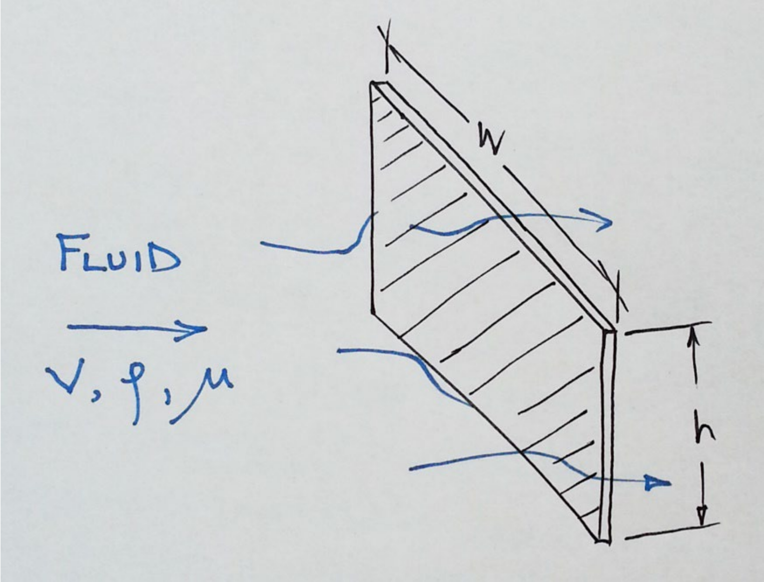


The drag force is a function of the flow velocity (V) and fluid properties (ρ and μ) and the dimensions of the rectangular plate. The plate has height h and width w.

Determine the dimensionless parameters (Pi terms) needed to conduct the experiment.

(This is a variation on the example that was mentioned in Part 1)

Example: Aerodynamic Drag on a Plate



Step 1: List the n variables

In this case you are told in the problem statement:

$$F_D = f_1(w, h, V, \mu, \rho)$$

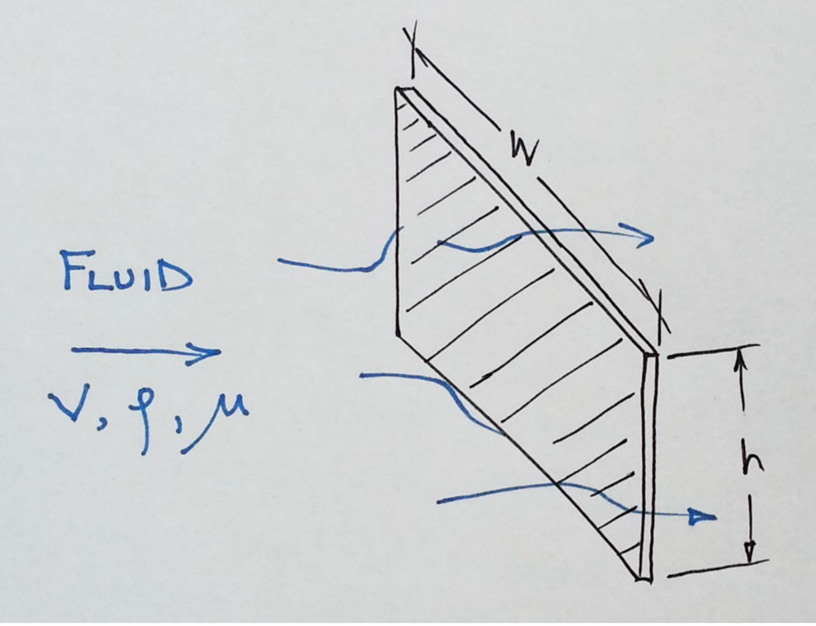
So, we have $n = 6$ variables. (Don't forget to count the dependent variable, F_D)

Why didn't we include the plate area, A, as a variable?

Because $A = h\,w$ is not independent of h and w.

✓ Check that all parameters are independent.

Example: Aerodynamic Drag on a Plate

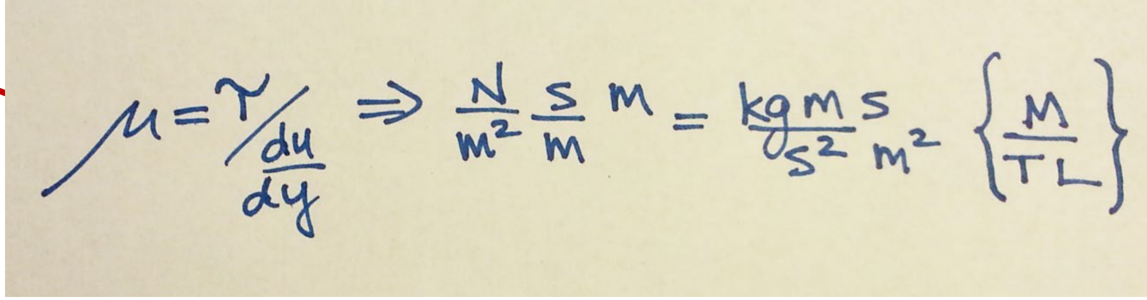


Step 2: Express each of the variables in terms of basic dimensions.

- We will use (M, L, T, Θ) for this example.

$\{F_D\} = \left\{\frac{ML}{T^2}\right\}$ $\{w\} = \{L\}$ $\{h\} = \{L\}$

$\{\rho\} = \left\{\frac{M}{L^3}\right\}$ $\{\mu\} = \left\{\frac{M}{LT}\right\}$ $\{V\} = \left\{\frac{L}{T}\right\}$



(No need to memorize. But you should know Newton's Law of Viscosity)

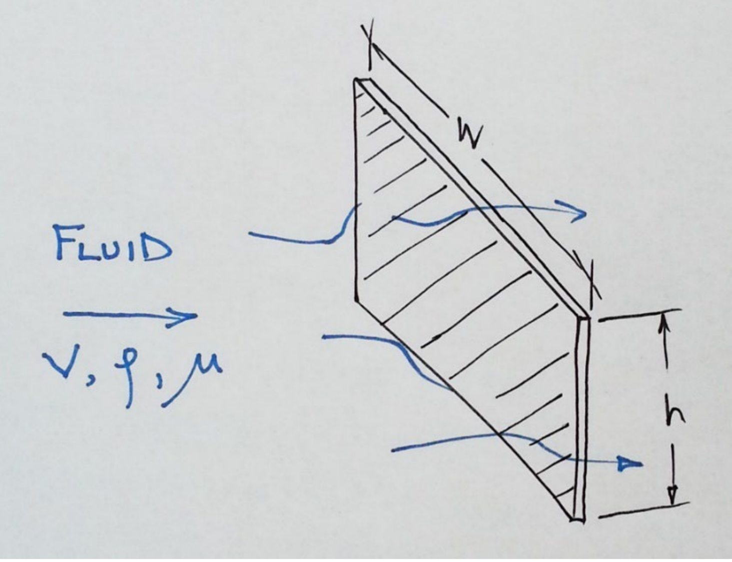
Step 3: Determine the number of Π parameters

We have $j = 3$ basic dimensions: M,L,T

We have $n = 6$ variables

∴ Buckingham Pi Theorem says we have $k = n - j = 6 - 3 = 3$ dimensionless Pi terms.

Example: Aerodynamic Drag on a Plate



$F_D = f_1(w, h, V, \mu, \rho)$

Step 4: Select "$j = 3$" repeating variables from the "$n = 6$" variables

Rules:

- Do not pick the dependent parameter (F_D) as a "repeater"
- All reference dimensions must be included in the "repeaters"
- The repeating variables cannot themselves form a dimensionless product

We select 3 repeating variable (your choice, with above restrictions): w, V, ρ (other choices are possible)

Let's check the conditions above, given: $\{w\} = \{L\}$ $\{V\} = \left\{\frac{L}{T}\right\}$ $\{\rho\} = \left\{\frac{M}{L^3}\right\}$

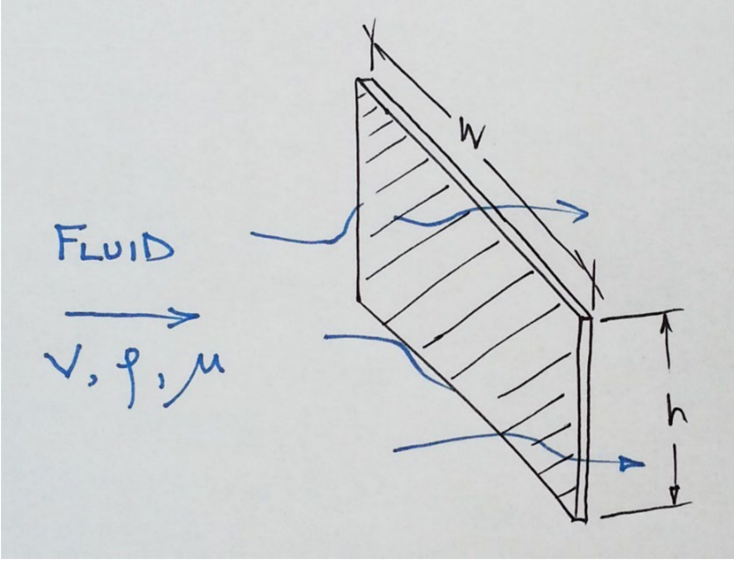
✓ Contain M, L and T

match exponents, left = right

✓ Do not form dimensionless term: $w^a V^b \rho^c = (L)^a (\frac{L}{T})^b (\frac{M}{L^3})^c = L^0 M^0 T^0 \rightarrow c = 0, b = 0, a = 0$

(∴ non-zero values of a, b and c do not exist that would produce a dimensionless term)

Example: Aerodynamic Drag on a Plate



$$F_D = f_1(w, h, V, \mu, \rho)$$

Step 5: Form $k = n - j = 6 - 3 = 3$ Pi terms

- Using the three non-repeating variables:

F_D h μ

$$\Pi_1 = F_D w^{a1} V^{b1} \rho^{c1} = \left(\frac{ML}{T^2}\right)(L)^{a1}\left(\frac{L}{T}\right)^{b1}\left(\frac{M}{L^3}\right)^{c1} = L^0 M^0 T^0$$

$$\Pi_2 = h\, w^{a2} V^{b2} \rho^{c2} = (L)(L)^{a2}\left(\frac{L}{T}\right)^{b2}\left(\frac{M}{L^3}\right)^{c2} = L^0 M^0 T^0$$

$$\Pi_3 = \mu\, w^{a3} V^{b3} \rho^{c3} = \left(\frac{M}{LT}\right)(L)^{a3}\left(\frac{L}{T}\right)^{b3}\left(\frac{M}{L^3}\right)^{c3} = L^0 M^0 T^0$$

repeating variables

- We now find the exponents $a1, b1, c1$ that make Π_1 dimensionless. Repeat for Π_2 and Π_3.

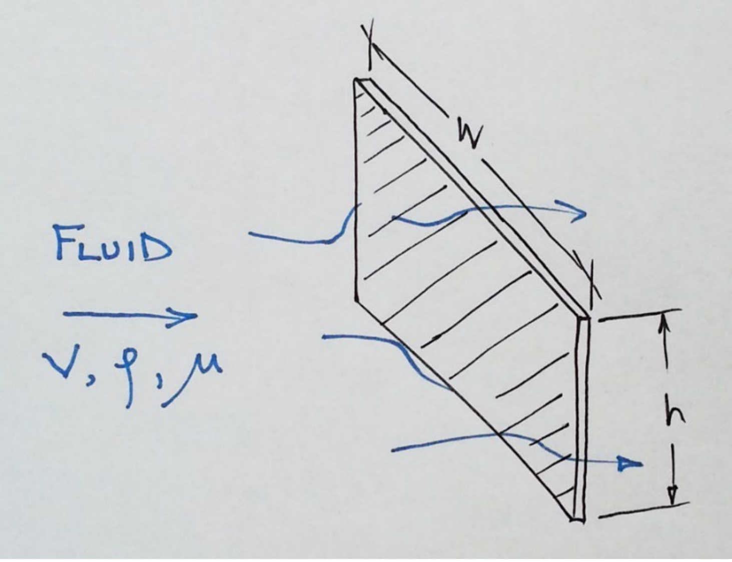
Example: Aerodynamic Drag on a Plate

Step 5: Form the Pi terms

- We proceed one at a time, using the three non-repeating variables: F_D

match exponents, left = right

$$\Pi_1 = F_D w^a V^b \rho^c = \left(\frac{ML}{T^2}\right)(L)^a\left(\frac{L}{T}\right)^b\left(\frac{M}{L^3}\right)^c = L^0 M^0 T^0$$



$$F_D = f_1(w, h, V, \mu, \rho)$$

We can now find the values of a, b and c that will make Π_1 dimensionless:

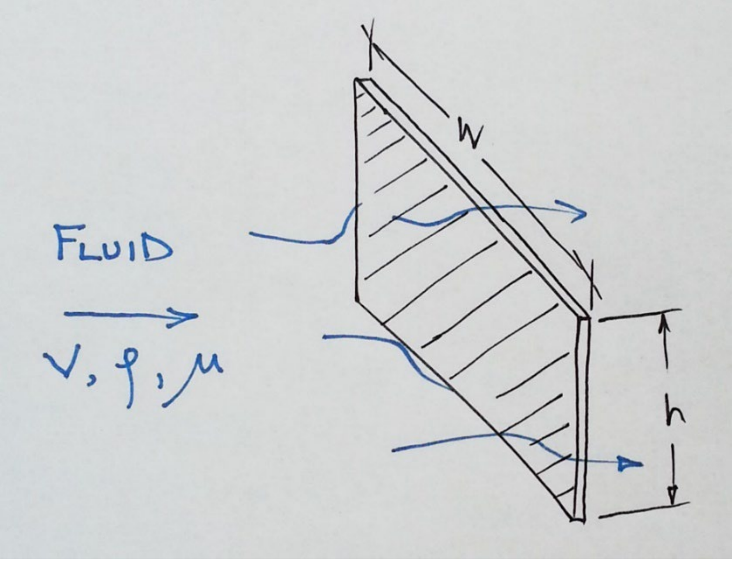
Exponents for M: $1 + c = 0$ $\quad \therefore c = -1$

Exponents for T: $-2 - b = 0$ $\quad \therefore b = -2$

Exponents for L: $1 + a + b - 3c = 0 \rightarrow a = 3c - b - 1$ $\quad \therefore a = -2$

So, $\Pi_1 = F_D w^{-2} V^{-2} \rho^{-1}$ $\qquad \Pi_1 = \dfrac{F_D}{w^2\, V^2\, \rho}$ ✓ dimensionless

Example: Aerodynamic Drag on a Plate



$F_D = f_1(w, h, V, \mu, \rho)$

Step 5: Form the Pi terms

- We proceed one at a time, using the three non-repeating variables: h

$$\Pi_2 = h\, w^a V^b \rho^c = (L)(L)^a(\frac{L}{T})^b(\frac{M}{L^3})^c = L^0 M^0 T^0$$

Again, we find the values of a, b and c that will make Π_2 have no dimensions:

Exponents for M: $c = 0$

Exponents for T: $-b = 0$

Exponents for L: $1 + a + b - 3c = 0 \rightarrow a = -1$

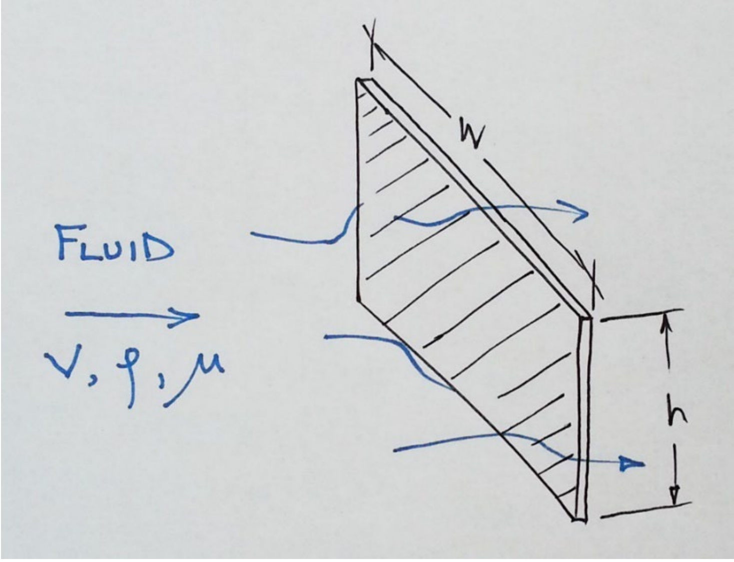
So, $\Pi_2 = hw^{-1}V^0\rho^0$ $\quad\quad \Pi_2 = \frac{h}{w}$ ✓ dimensionless

Example: Aerodynamic Drag on a Plate

Step 5: Form the Pi terms

- We proceed one at a time, using the three non-repeating variables: μ

$$\Pi_3 = \mu\, w^a V^b \rho^c = (\frac{M}{LT})(L)^a(\frac{L}{T})^b(\frac{M}{L^3})^c = L^0 M^0 T^0$$



$$F_D = f_1(w, h, V, \mu, \rho)$$

Again, we find the values of a, b and c that will make Π_3 have no dimensions:

Exponents for M: $1 + c = 0$ $\quad \therefore c = -1$

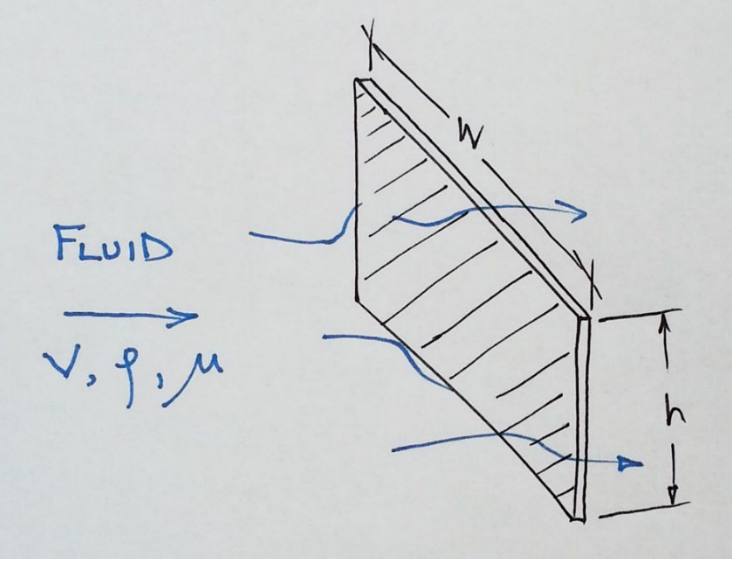
Exponents for T: $-1 - b = 0$ $\quad \therefore b = -1$

Exponents for L: $-1 + a + b - 3c = 0 \rightarrow a = 3c - b + 1$ $\quad \therefore a = -1$

So, $\Pi_3 = \mu\, w^{-1} V^{-1} \rho^{-1}$ $\qquad \Pi_3 = \frac{\mu}{wV\rho}$

✓ dimensionless
(recognize as 1/Re)

Example: Aerodynamic Drag on a Plate



$F_D = f_1(w, h, V, \mu, \rho)$

Step 6: Express the final form: $\Pi_1 = f_2(\Pi_2, \Pi_3)$

- Recall: Put the dependent variable in the numerator of Π_1

- We have shown:

$$\Pi_1 = \frac{F_D}{w^2 V^2 \rho}, \quad \Pi_2 = \frac{h}{w}, \Pi_3 = \frac{\mu}{wV\rho}$$

So, $\frac{F_D}{w^2 \rho V^2} = f_2\left(\frac{h}{w}, \frac{\mu}{wV\rho}\right)$ ← $\frac{1}{Re}$

Any Pi parameter can be inverted, since the functional relationship is unknown. So, we can write:

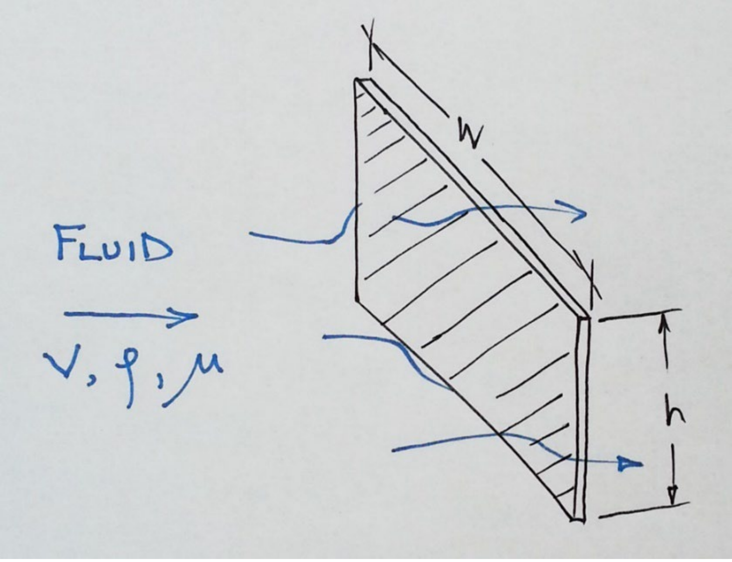
$$\frac{F_D}{w^2 \rho V^2} = f_3\left(\frac{h}{w}, \frac{\rho V w}{\mu}\right)$$

Example: Aerodynamic Drag on a Plate

- Rewriting:

$$\frac{F_D}{w^2\,\rho\,V^2} = f_3\left(\frac{h}{w},\ \frac{\rho V w}{\mu}\right)$$

Dimensionless drag force — Geometry (aspect ratio) — Reynolds number



- This is as far as dimensional analysis can go. The unknown function f_3 ***cannot*** be determined by dimensional analysis.
- Experiments can be conducted to find this function (with greatly reduced effort).

<u>Note</u>

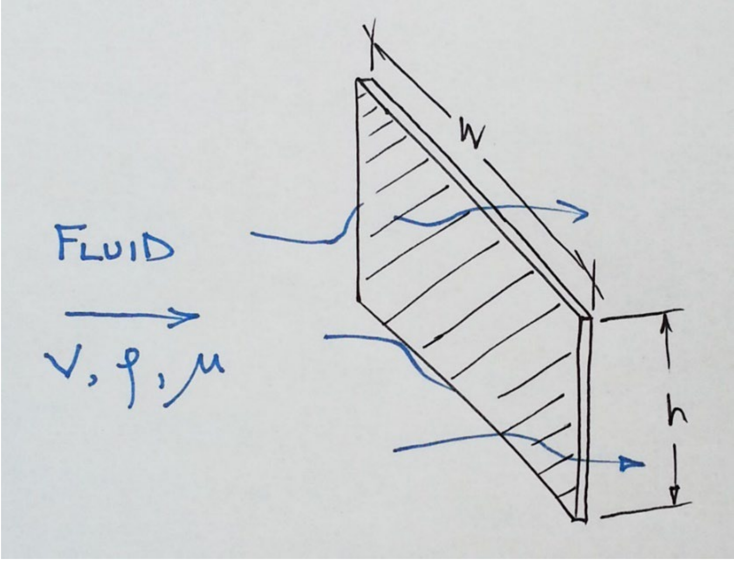
- These Pi parameters are not unique. They depend upon the initial choice of “repeaters”. If you pick different repeating variables, you will get different Pi parameters.

Example: Aerodynamic Drag on a Plate

- Rewriting:

$$\frac{F_D}{w^2\,\rho\,V^2} = f_3\left(\frac{h}{w}, \frac{\rho V w}{\mu}\right)$$

Dimensionless drag force | Geometry (aspect ratio) | Reynolds number



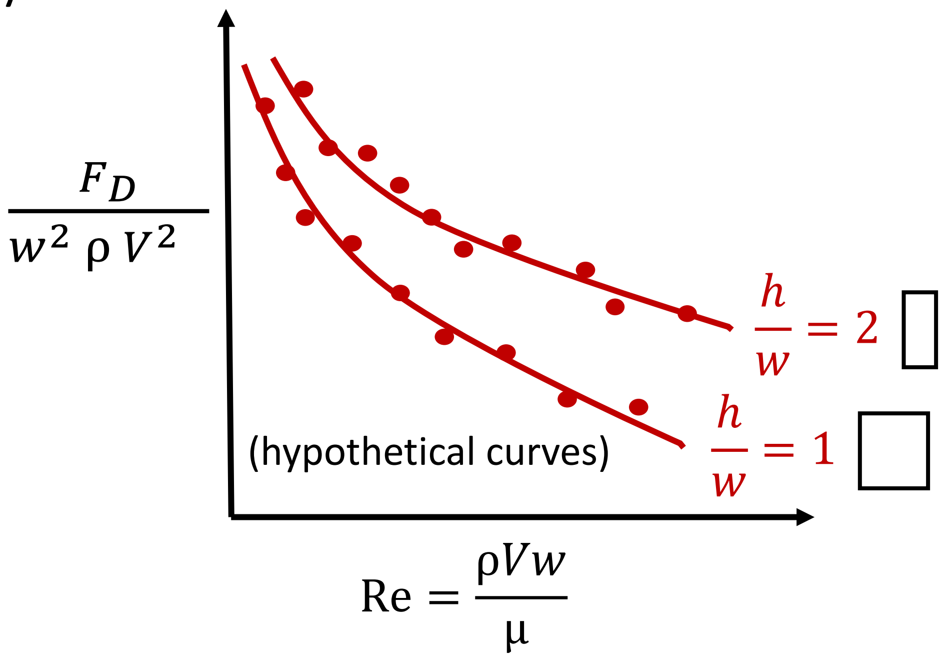
$$F_D = f_1(w, h, V, \mu, \rho)$$

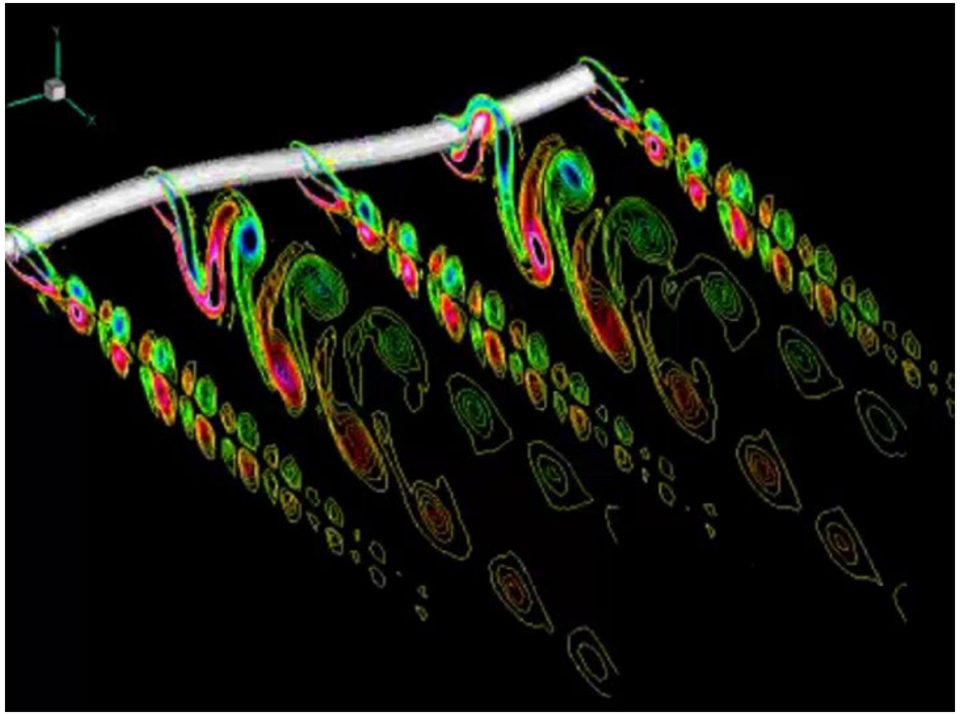
- This result stems from the principle of dimensional homogeneity (PDH). Same general result as introduced in Part 1.

- Reduction from 6 dimensional to 3 dimensionless variables.

- For fixed geometry (h/w), the dimensionless drag forces is only a function of Reynolds number.

- Thus, only a single set of experiments in a single fluid is needed for each aspect ratio!





CFD simulation of flow-induced vibration of a cable.

END NOTES

Presentation prepared and delivered by Dr. David Naylor.

MEC516/BME516: Fluid Mechanics I

The Buckingham Pi Theorem

RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering