

MEC516/BME516
Fluid Mechanics I

Chapter 4
Recommended Problem Set Solutions

1. This unsteady continuity equations is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

The flow is one-dimensional, $v = w = 0$ everywhere. Noting that the gas density ρ is not a function of x , the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) = -\frac{\partial}{\partial x}\left(-\rho \frac{x}{L} V_p\right) = \rho \frac{V_p}{L}$$

Using the ideal gas equation of state: Table A.4, For carbon dioxide $R=189 \text{ J/kgK}$

$$\rho = \frac{p}{RT} = \frac{(150,000 + 98,000) \frac{\text{N}}{\text{m}^2}}{189 \frac{\text{Nm}}{\text{kgK}} (50 + 273) \text{K}} = 4.06 \frac{\text{kg}}{\text{m}^3}$$

Thus, the instantaneous rate of change of gas density is:

$$\frac{\partial \rho}{\partial t} = \rho \frac{V_p}{L} = 4.06 \frac{\text{kg}}{\text{m}^3} \left(\frac{1.5 \frac{\text{m}}{\text{s}}}{0.15 \text{ m}} \right) = 40.6 \frac{\text{kg}}{\text{m}^3 \text{ s}}$$

2. (a) The flow is unsteady because time t appears x - and y -components of velocity, i.e., the velocity field is changing with time. The flow is three-dimensional because all three velocity components (u , v , w) are nonzero and depend upon the three spatial coordinates.
(b) Evaluating the three components of the acceleration vector:

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 4x + 4tx(4t) - 2t^2y(0) + 4xz(0) = 4x + 16t^2x \\ a_y &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -4ty + 4tx(0) - 2t^2y(-2t^2) + 4xz(0) = -4ty + 4t^4y \\ a_z &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = 0 + 4tx(4z) - 2t^2y(0) + 4xz(4x) = 16txz + 16x^2z \end{aligned}$$

The fluid acceleration vector is:

$$\mathbf{a} = (4x + 16t^2x)\mathbf{i} + (-4ty + 4t^4y)\mathbf{j} + (16txz + 16x^2z)\mathbf{k}$$

$$\text{At } (x, y, z) = (-1, 1, 0) \quad \mathbf{a} = (-4 - 16t^2)\mathbf{i} + (-4t + 4t^4)\mathbf{j} + 0 \mathbf{k}$$

3. This is a steady flow ($\frac{\partial u}{\partial t} = 0$) one-dimensional flow:

$$a_x = u \frac{\partial u}{\partial x} = V_o \left(1 + \frac{2x}{L}\right) \frac{2V_o}{L} = \frac{2V_o^2}{L} \left(1 + \frac{2x}{L}\right)$$

For the case $V_o = 10$ ft/s and $L = 6.0$ in:

At the nozzle inlet ($x=0$):

$$a_x = \frac{2V_o^2}{L} = \frac{2 \left(10 \frac{ft}{s}\right)^2}{0.5 ft} = 400 \frac{ft}{s^2}$$

At the nozzle exit ($x=L=0.5$ ft):

$$a_x = \frac{2V_o^2}{L} \left(1 + \frac{2x}{L}\right) = \frac{2 \left(10 \frac{ft}{s}\right)^2}{0.5 ft} (1 + 2) = 1200 \frac{ft}{s^2}$$

This illustrates a key concept: A steady flow can have an acceleration. It is called “convective acceleration”, produced by fluid being swept into a region of different velocity. In this case, the fluid accelerates as it passes through the converging nozzle, even though the volume flow rate is not changing with time.

4. An *inviscid* flow is an idealized flow with no viscous effects (i.e., $\mu=0$). All fluids have viscosity. However, some real-world flows can be closely approximate as inviscid. For steady 2-D flow of an incompressible inviscid fluid, the x-momentum equation becomes:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x$$

Substituting the velocity field into the momentum equation:

$$\rho(2xy(2y) - y^2(2x)) = -\frac{\partial p}{\partial x} + \rho g_x$$

$$\frac{\partial p}{\partial x} = -2xy^2\rho + \rho g_x$$

5. (a) This 2-D flow field satisfies the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad 0 + 0 = 0$$

(b) All real flows must also satisfy the conservation of momentum i.e., must satisfy the Navier-Stokes equations (conservation of momentum):

x- momentum equation:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x \quad \rho(4y(0) + 2x(4)) = -\frac{\partial p}{\partial x} + (0) + (0)$$

$$\frac{\partial p}{\partial x} = -8\rho x$$

y-momentum equation:

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y \quad \rho(4y(2) + 2x(0)) = -\frac{\partial p}{\partial y} + (0) + (0)$$

$$\frac{\partial p}{\partial y} = -8\rho y$$

For a real flow that satisfies the Navier-Stokes equations $p(x,y)$ is a single function. Thus, these expressions for $\partial p/\partial x$ and $\partial p/\partial y$ must give the same result if we take the cross derivative: $\partial^2 p/\partial x \partial y$

This mathematical “consistency check” demonstrates that a single pressure field $p(x,y)$ exists that allows the x- and y- momentum equations to be conserved. For the current flow:

$$\frac{\partial}{\partial y} \left(\frac{\partial p}{\partial x} \right) = \frac{\partial}{\partial y} (-8\rho x) = 0$$

$$\frac{\partial}{\partial x} \left(\frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial x} (-8\rho y) = 0$$

Both cross derivatives give the same result. This demonstrates that the flow field satisfies the conservation of momentum. It is a possible real-world flow field.

The pressure field is found by integration:

$$\frac{\partial p}{\partial x} = -8\rho x \quad p = -4\rho x^2 + f(y) + C$$

$$\frac{\partial p}{\partial y} = -8\rho y \quad p = -4\rho y^2 + f(x) + C$$

(Why is $f(y)$ added to the right-hand side when integrating with respect to x ? You must add $f(y)$ because pressure is a function of x and y . Consider the reverse operation which is partial differentiation with respect to x . In that case $f(y)$ is treated as a constant.)

Comparing these two equations we can see that: $f(x) = -4\rho x^2$ and $f(y) = -4\rho y^2$
Hence:

$$p = -4\rho y^2 - 4\rho x^2 + C$$

The constant can be evaluated using pressure at the origin: At $x=y=0$, $p = p_o \rightarrow C = p_o$
The pressure field is:

$$p = -4\rho y^2 - 4\rho x^2 + p_o$$

6. Note that the x-component of gravity is: $g_x = g \sin \theta$. Also, the pressure along a free surface is constant and equal to atmospheric pressure. Thus, pressure does not change in the x-direction i.e., $\partial p/\partial x=0$

Conservation of x-momentum requires:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x$$

Making the substitutions:

$$\rho (Cy(2h - y)(0) + 0(2Ch - 2Cy)) = 0 + \mu(0 - 2C) + \rho g \sin \theta$$

$$C = \frac{\rho g \sin \theta}{2\mu}$$

(a) The volume flow rate is found by integrating the velocity profile over the thickness of the fluid layer:

$$Q = \int_0^h u \, dy = \frac{\rho g \sin \theta}{2\mu} \int_0^h (2hy - y^2) \, dy$$

$$Q = \frac{\rho g \sin \theta}{2\mu} \left[hy^2 - \frac{y^3}{3} \right]_{y=0}^{y=h} = \frac{\rho g \sin \theta}{3\mu} h^3$$

7. (a) The exact solution for laminar flow between parallel plates was derived in the lecture presentations. It can also be found in Chapter 4 of the textbook. This solution shows that the velocity distribution is parabolic with an average velocity:

$$u_{avg} = -\frac{h^2}{3\mu} \frac{dp}{dx}$$

Note from the solution that h is the half-width of the channel, $h=4.0\text{mm}$.

Table A.3: SAE 10W oil $\mu=0.104 \text{Ns/m}^2$, $\rho_o = 870 \frac{\text{kg}}{\text{m}^3}$ $\gamma_o = 8535 \frac{\text{N}}{\text{m}^3}$

Note that pressure decreases in the flow direction due to head losses. The pressure drop between the manometer taps is:

$$\Delta p = (\gamma_o - \gamma_m) \Delta h_m = (8535 - 132900) \frac{\text{N}}{\text{m}^3} (0.06\text{m}) = -7462 \text{ Pa}$$

The pressure gradient is constant in a fully developed flow:

$$\frac{dp}{dx} = \frac{\Delta p}{L} = \frac{-7462 \text{ Pa}}{1.0\text{m}} = -7462 \frac{\text{Pa}}{\text{m}}$$

Making the substitutions:

$$u_{avg} = -\frac{h^2}{3\mu} \frac{dp}{dx} = -\frac{(0.004\text{m})^2}{3 \left(0.104 \frac{\text{Ns}}{\text{m}^2}\right)} \left(-7462 \frac{\text{Pa}}{\text{m}}\right) = 0.383 \frac{\text{m}}{\text{s}}$$

The volume flow rate per unit width (into the page):

$$Q = u_{avg} (2h) = 0.383 \frac{\text{m}}{\text{s}} (0.08\text{m}) = 0.00306 \frac{\text{m}^3}{\text{s}}$$

- (b) The hydraulic diameter is the equivalent diameter used to calculate the Reynolds number for a non-circular duct. You will need it for Lab 4. The hydraulic diameter is four times the cross-sectional

area divided by the wetted perimeter. For a rectangular channel of height $2h$ and width w (into the page):

$$D_h = \frac{4A_c}{P_{wet}} = \frac{4(2h)w}{2(2h) + 2w}$$

In this two-dimensional flow problem. The channel width is much great than its height $w \gg 2h$. So, the hydraulic diameter is:

$$D_h = \frac{4(2h)w}{2w} = 4h = 4(4mm) = 16mm$$

The Reynolds number based on the hydraulic diameter is:

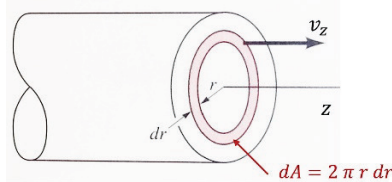
$$Re = \frac{\rho D_h u_{avg}}{\mu} = \frac{870 \frac{kg}{m^3} (0.016 m) 0.383 \frac{m}{s}}{0.104 \frac{kg}{m \cdot s}} = 51.3$$

This confirms that the flow will be laminar since $Re < 2300$.

8. Note the $Q = \bar{V}A$. The volume flow rate is obtained by integrating the local velocity with respect to area:

$$Q = \int_0^R v_z dA$$

Note that the differential area is an annular area: $dA = 2\pi r dr$.



The volume flow rate can be expressed as:

$$Q = \int_0^R V_{max} \left(1 - \frac{r^2}{R^2}\right) 2\pi r dr = 2\pi V_{max} \int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

Integrating gives:

$$Q = 2\pi V_{max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R = 2\pi V_{max} R^2 \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{1}{2} \pi V_{max} R^2$$

Using the fact that $Q = \bar{V}A$ gives:

$$\bar{V}A = \bar{V}\pi R^2 = \frac{1}{2} \pi V_{max} R^2$$

$$\bar{V} = \frac{1}{2} V_{max}$$