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MEC516/BME516
Fluid Mechanics I

Chapter 4
Recommended Problem Set Solutions

1. This unsteady continuity equations is:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

The flow is one-dimensional, $v=w=0$ everywhere. Noting that the gas density ρ is not a function of x, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t}=-\frac{\partial}{\partial x}(\rho u)=-\frac{\partial}{\partial x}\left(-\rho\frac{x}{L}V_p\right)=\rho\frac{V_p}{L}$$

Using the ideal gas equation of state: Table A.4, For carbon dioxide R=189 J/kgK

$$\rho=\frac{p}{RT}=\frac{(150{,}000+98{,}000)\frac{N}{m^2}}{189\frac{Nm}{kgK}(50+273)K}=4.06\frac{kg}{m^3}$$

Thus, the instantaneous rate of change of gas density is:

$$\frac{\partial \rho}{\partial t}=\rho\frac{V_p}{L}=4.06\frac{kg}{m^3}\left(\frac{1.5\frac{m}{s}}{0.15\ m}\right)=40.6\ \frac{kg}{m^3 s}$$

2. (a) The flow is unsteady because time t appears x- and y-components of velocity, i.e., the velocity field is changing with time. The flow is three-dimensional because all three velocity components $(u,\ v,\ w)$ are nonzero and depend upon the three spatial coordinates.
(b) Evaluating the three components of the acceleration vector:

$$a_x=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=4x+4tx(4t)-2t^2y(0)+4xz(0)=4x+16t^2x$$
$$a_y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-4ty+4tx(0)-2t^2y(-2t^2)+4xz(0)=-4ty+4t^4y$$
$$a_z=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0+4tx(4z)-2t^2y(0)+4xz(4x)=16txz+16x^2z$$

The fluid acceleration vector is:

$$\boldsymbol{a}=(4x+16t^2x)\boldsymbol{i}+(-4ty+4t^4y)\boldsymbol{j}+(16txz+16x^2z)\boldsymbol{k}$$

At $(x,y,z)=(-1,1,0)$ $\quad \boldsymbol{a}=(-4-16t^2)\boldsymbol{i}+(-4t+4t^4)\boldsymbol{j}+0\ \boldsymbol{k}$

3. This is a steady flow ($\frac{\partial u}{\partial t} = 0$) one-dimensional flow:

$$a_x = u\frac{\partial u}{\partial x} = V_o\left(1 + \frac{2x}{L}\right)\ \frac{2V_o}{L} = \frac{2V_o^2}{L}\left(1 + \frac{2x}{L}\right)$$

For the case $V_o = 10$ ft/s and L = 6.0 in:

At the nozzle inlet (x=0):

$$a_x = \frac{2V_o^2}{L} = \frac{2\left(10\frac{ft}{s}\right)^2}{0.5\,ft} = 400\frac{ft}{s^2}$$

At the nozzle exit (x=L=0.5 ft):

$$a_x = \frac{2V_o^2}{L}\left(1 + \frac{2x}{L}\right) = \frac{2\left(10\frac{ft}{s}\right)^2}{0.5\,ft}(1 + 2) = 1200\frac{ft}{s^2}$$

This illustrates a key concept: A steady flow can have an acceleration. It is called “convective acceleration”, produced by fluid being swept into a region of different velocity. In this case, the fluid accelerates as it passes through the converging nozzle, even though the volume flow rate is not changing with time.

4. An *inviscid* flow is an idealized flow with no viscous effects (i.e., μ=0). All fluids have viscosity. However, some real-world flows can be closely approximate as inviscid. For steady 2-D flow of an incompressible inviscid fluid, the x-momentum equation becomes:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x$$

Substituting the velocity field into the momentum equation:

$$\rho(2xy(2y) - y^2(2x)) = -\frac{\partial p}{\partial x} + \rho g_x$$

$$\frac{\partial p}{\partial x} = -2xy^2\rho + \rho g_x$$

5. (a) This 2-D flow field satisfies the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 0 + 0 = 0$$

(b) All real flows must also satisfy the conservation of momentum i.e., must satisfy the Navier-Stokes equations (conservation of momentum):

x- momentum equation:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x \qquad \rho\big(4y(0) + 2x(4)\big) = -\frac{\partial p}{\partial x} + (0) + (0)$$

$$\frac{\partial p}{\partial x} = -8\rho x$$

y-momentum equation:

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\rho g_y \quad \rho\big(4y(2)+2x(0)\big)=-\frac{\partial p}{\partial y}+(0)+(0)$$

$$\frac{\partial p}{\partial y}=-8\rho y$$

For a real flow that satisfies the Navier-Stokes equations p(x,y) is a single function. Thus, these expressions for ∂p/∂x and ∂p/∂y must give the same result if we take the cross derivative: $\partial^2 p/\partial x\partial y$

This mathematical "consistency check" demonstrates that a single pressure field p(x,y) exists that allows the x- and y- momentum equations to be conserved. For the current flow:

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)=\frac{\partial}{\partial y}(-8\rho x)=0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)=\frac{\partial}{\partial x}(-8\rho y)=0$$

Both cross derivatives give the same result. This demonstrates that the flow field satisfies the conservation of momentum. It is a possible real-world flow field.

The pressure field is found by integration:

$$\frac{\partial p}{\partial x}=-8\rho x \quad p=-4\rho x^2+f(y)+C$$
$$\frac{\partial p}{\partial y}=-8\rho y \quad p=-4\rho y^2+f(x)+C$$

(Why is f(y) added to the right-hand side when integrating with respect to x? You must add f(y) because pressure is a function of x and y. Consider the reverse operation which is <u>*partial*</u> differentiation with respect to x. In that case f(y) is treated as a constant.)

Comparing these two equations we can see that: $f(x)=-4\rho x^2$ and $f(y)=-4\rho y^2$
Hence:

$$p=-4\rho y^2-4\rho x^2+C$$

The constant can be evaluated using pressure at the origin: At x=y=0, $p=p_o$ → $C=p_o$
The pressure field is:

$$p=-4\rho y^2-4\rho x^2+p_o$$

6. Note that the x-component of gravity is: $g_x=g\sin\theta$. Also, the pressure along a free surface is constant and equal to atmospheric pressure. Thus, pressure does not change in the x-direction i.e., ∂p/∂x=0

Conservation of x-momentum requires:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho g_x$$

Making the substitutions:

$$\rho\,(\,Cy(2h-y)(0)\ +0(2Ch-2Cy)) = 0+\mu(\,0-2C)+\ \rho g\sin\theta$$

$$C=\frac{\rho g\sin\theta}{2\mu}$$

(a) The volume flow rate is found by integrating the velocity profile over the thickness of the fluid layer:

$$Q=\int_0^h u\,dy=\frac{\rho g\sin\theta}{2\mu}\int_0^h (2hy-y^2)\,dy$$

$$Q=\frac{\rho g\sin\theta}{2\mu}\left[hy^2-\frac{y^3}{3}\right]_{y=0}^{y=h}=\frac{\rho g\sin\theta\ h^3}{3\mu}$$

7. (a) The exact solution for laminar flow between parallel plates was derived in the lecture presentations. It can also be found in Chapter 4 of the textbook. This solution shows that the velocity distribution is parabolic with an average velocity:

$$u_{avg}=-\frac{h^2}{3\mu}\frac{dp}{dx}$$

Note from the solution that *h* is the half-width of the channel, *h*=4.0mm.

Table A.3: SAE 10W oil μ=0.104 Ns/m^2, $\rho_o=870\frac{kg}{m^3}\ \ \gamma_o=8535\frac{N}{m^3}$

Note that pressure decreases in the flow direction due to head losses. The pressure drop between the manometer taps is:

$$\Delta p=(\gamma_o-\gamma_m)\,\Delta h_m=(8535-132900)\frac{N}{m^3}(0.06m)=-7462\,Pa$$

The pressure gradient is constant in a fully developed flow:

$$\frac{dp}{dx}=\frac{\Delta p}{L}=\frac{-7462\,Pa}{1.0m}=-7462\frac{Pa}{m}$$

Making the substitutions:

$$u_{avg}=-\frac{h^2}{3\mu}\frac{dp}{dx}=-\frac{(0.004m)^2}{3\left(0.104\frac{\mathrm{Ns}}{\mathrm{m}^2}\right)}\left(-7462\frac{Pa}{m}\right)=0.383\frac{m}{s}$$

The volume flow rate per unit width (into the page):

$$Q=u_{avg}\,(2h)=0.383\frac{m}{s}(0.08m)=0.00306\frac{m^3}{s}$$

(b) The hydraulic diameter is the equivalent diameter used to calculate the Reynolds number for a non-circular duct. You will need it for Lab 4. The hydraulic diameter is four times the cross-sectional area divided by the wetted perimeter. For a rectangular channel of height 2h and width w (into the page):

$$D_h = \frac{4A_c}{P_{wet}} = \frac{4(2h)w}{2(2h) + 2w}$$

In this two-dimensional flow problem. The channel width is much great than its height w>>2h. So, the hydraulic dimeter is:

$$D_h = \frac{4(2h)w}{2w} = 4h = 4(4mm) = 16mm$$

The Reynolds number based on the hydraulic diameter is:

$$Re = \frac{\rho D_h u_{avg}}{\mu} = \frac{870 \frac{kg}{m^3} (0.016\, m) 0.383 \frac{m}{s}}{0.104\, \frac{kg}{ms}} = 51.3$$

This confirms that the flow will be laminar.