MEC516/BME516: Fluid Mechanics I

Chapter 4: Differential Relations for Fluid Flow
Part 6

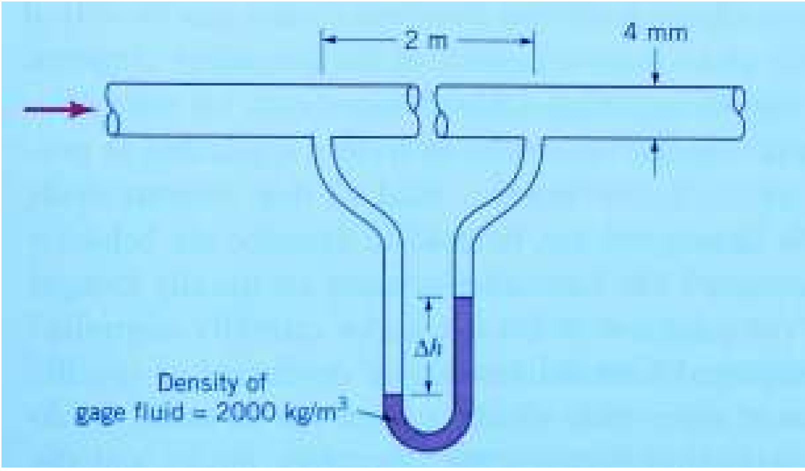
RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering

Overview

- **An Exact Solution to the Continuity and Navier-Stokes Equations**
  - Laminar incompressible flow in a round pipe (*Hagen-Poiseuille Flow*).
  - Solution in cylindrical coordinates: r, θ, z
  - "Poiseuille's Law" for flow in small tubes.
    (Motivated in part by Poiseuille's interest in blood flow through capillaries.)

- **Example**

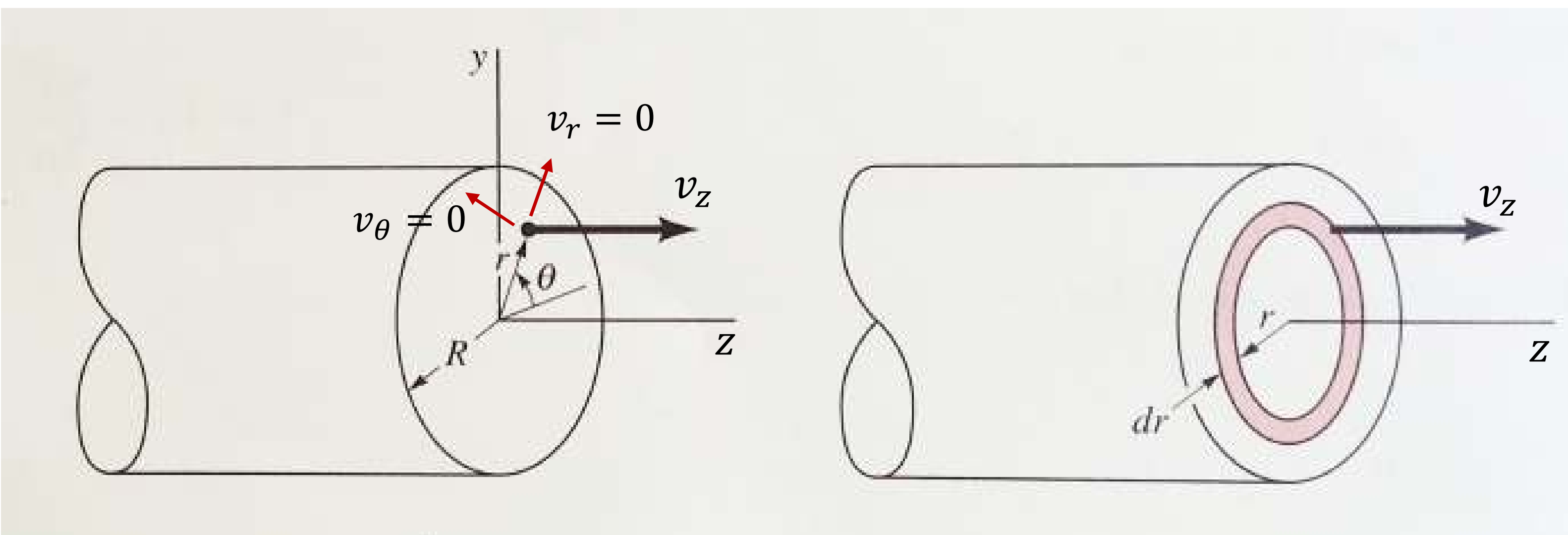
  Calculate the flow in a small tube using "Poiseuille's law". Reynolds number calculation.



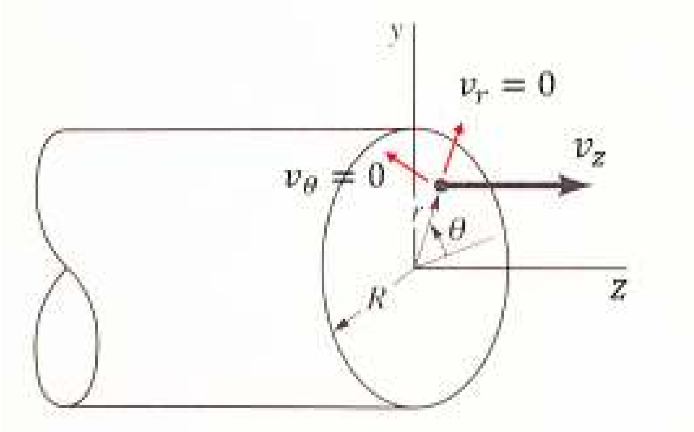
Steady Laminar Flow in a Circular Tube

Problem Definition

- Consider steady laminar incompressible viscous flow in a round pipe.
- Cylindrical coordinates $\boldsymbol{V} = \boldsymbol{V}(r, z, \theta) = v_r\,\mathbf{i} + v_\theta\,\mathbf{j} + v_z\,\mathbf{k}$
- The flow is far from the pipe entrance. So, the flow is purely axial: $v_r = v_\theta = 0$.
- Neglect gravity. (Adds hydrostatic pressure gradient; does not affect flow.)



Steady Laminar Flow in a Circular Tube



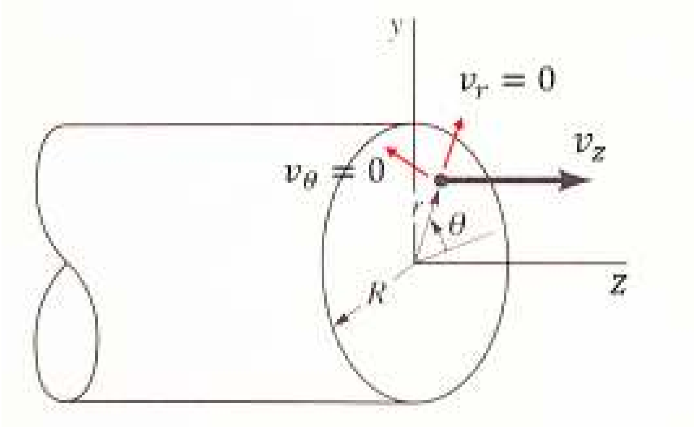
- Start with the incompressible continuity equation (conservation of mass)

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

$$\frac{\partial}{\partial z}(v_z) = 0$$

- Flow is fully developed.
- Velocity field and hence, the axial pressure gradient $(\frac{\partial p}{\partial z})$ does not change in the z direction.

Steady Laminar Flow in a Circular Tube



- Incompressible Navier-Stokes equation in the z-direction

Conservation of z-momentum:

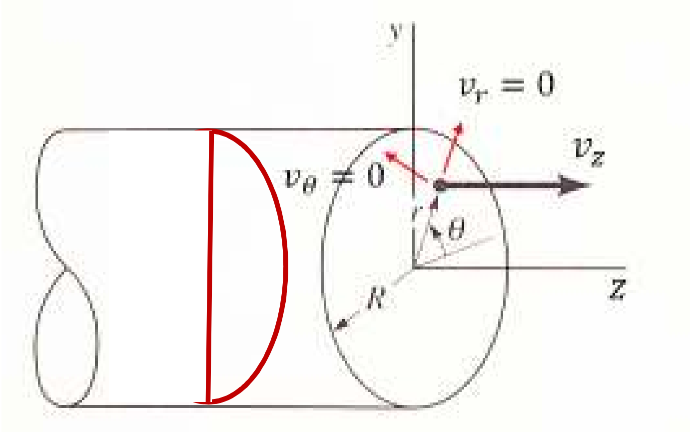
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right\}$$

steady $v_r = 0$ $v_\theta = 0$ no swirl   fully dev.   no variation in θ fully dev.

- Similarly, conservation of momentum in r and θ directions give: $\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0 \quad \therefore\ p = p(z)\ only$

- So, we get: $$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dp}{dz} = const < 0$$

Steady Laminar Flow in a Circular Tube



$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dp}{dz}$$

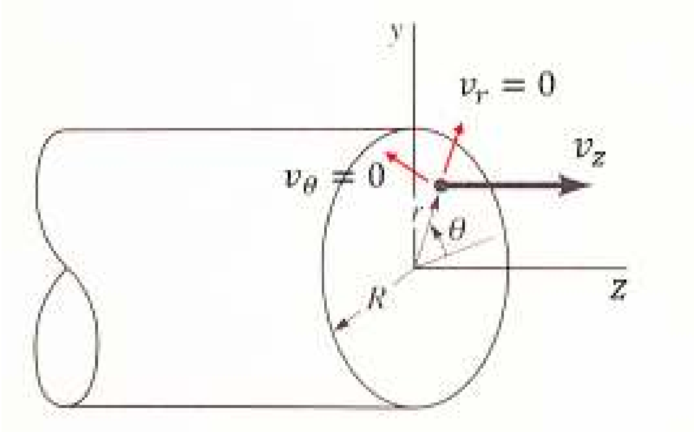
$$d\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz}\, r\, dr$$

- Integrating (noting that $\frac{dp}{dz} = const$): $\quad r\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r^2 + C_1 \quad \frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}\, r + \frac{C_1}{r}$

- The flow is symmetrical about the centre line ($r = 0$): $\left.\frac{dv_z}{dr}\right|_{r=0} = 0$ Thus, $C_1 = 0$

Thus: $$\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r$$

Steady Laminar Flow in a Circular Tube



$$\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r$$

- Integrating again: $v_z = \frac{1}{4\mu}\frac{dp}{dz}\ r^2 + C_2$

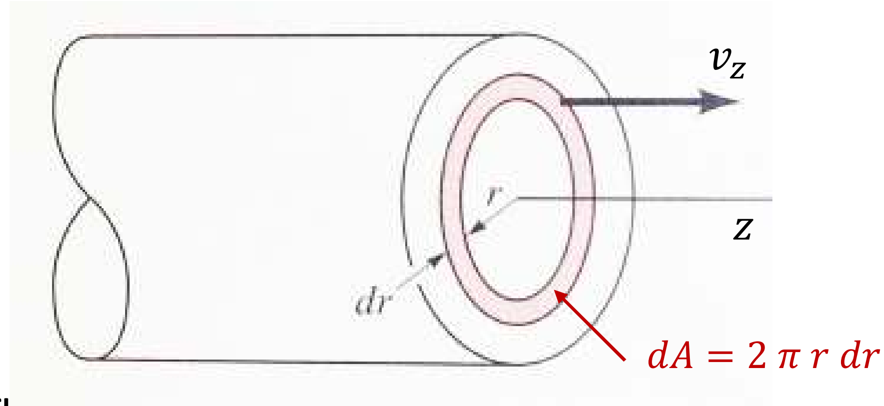
- Now we use the no slip boundary condition $r = R$ to evaluate C_2: $\quad v_z(R) = 0$

$$0 = \frac{1}{4\mu}\frac{dp}{dz}\ R^2 + C_2 \qquad \text{Thus,}\quad C_2 = -\frac{1}{4\mu}\frac{dp}{dz}\ R^2$$

- So, the velocity field becomes: $\boxed{v_z = -\frac{1}{4\mu}\frac{dp}{dz}\ (R^2 - r^2)}$ $\qquad v_{max} = -\frac{R^2}{4\mu}\frac{dp}{dz}$

- This is the classical solution for laminar fully developed flow in a round tube, called *Hagen-Poiseuille Flow.*

Steady Laminar Flow in a Circular Tube



The velocity field: $v_z = -\frac{1}{4\mu}\frac{dp}{dz}\,(R^2 - r^2)$

- We can integrate this velocity field to get the volume flow rate:

$$Q = \int_{r=0}^{r=R} v_z\,dA = \int_{r=0}^{r=R} -\frac{1}{4\mu}\frac{dp}{dz}\,(R^2 - r^2)\,2\pi\,r\,dr$$

- Result: $Q = -\frac{\pi R^4}{8\mu}\frac{dp}{dz}$

- For a pipe of length L with pressure drop Δp:

$$Q = -\frac{\pi R^4}{8\mu}\frac{\Delta p}{L}$$

Poiseuille's Law

Steady Laminar Flow in a Circular Tube

- For a pipe of length L with pressure drop Δp:

$$Q = -\frac{\pi R^4}{8\mu}\frac{\Delta p}{L}$$

Poiseuille's Law

Comments

- This result applies for laminar flow, Re<2300, where $Re = \frac{\rho \bar{V} D}{\mu}$
- The flow rate is surprisingly sensitive to the tube size! $Q \sim D^4$

<u>Medical Application</u>

People with asthma take bronchodialtors (drugs that expands the tiny air passages to the lungs). If the airway increases in diameter by say 20%, the increase in air flow will be $1.2^4 = 2.1$.

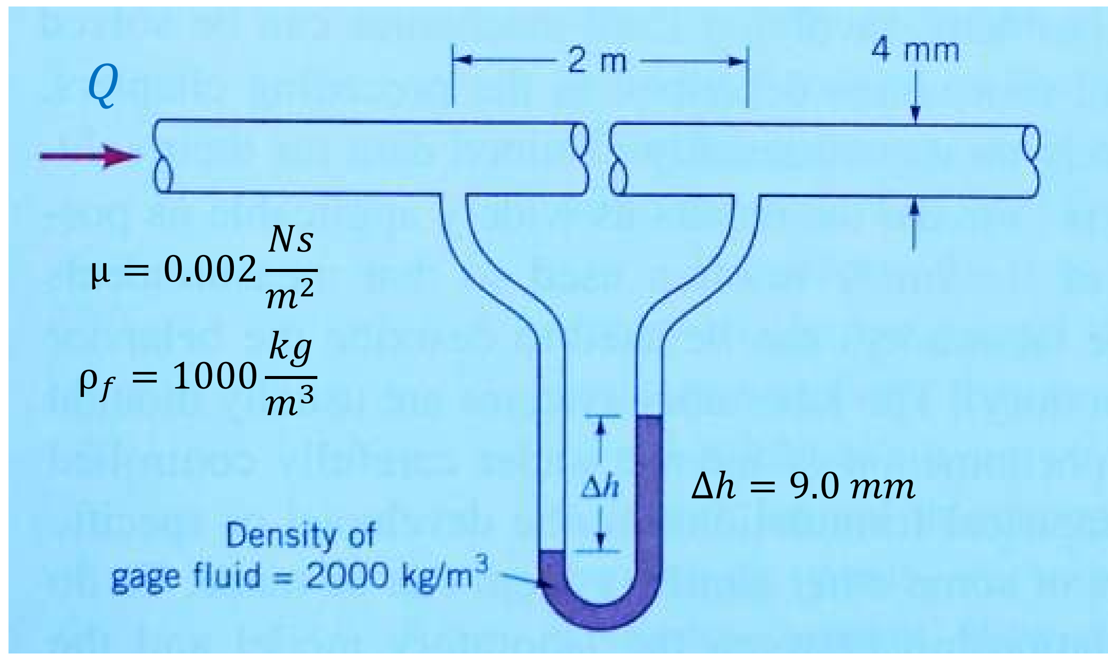
More than twice the air flow for the same breathing effort!

Example

A liquid with dynamic viscosity of $\mu = 0.002\ Ns/m^2$ and density $\rho_f = 1000\ kg/m^3$ flows at a steady rate in a tube with an inside diameter of $D = 4mm$. A U-tube manometer with a gage fluid with density $\rho_g = 2000\ kg/m^3$ is used to measure the pressure drop in the pipe. The manometer deflection is $\Delta h = 9.0\ mm$.

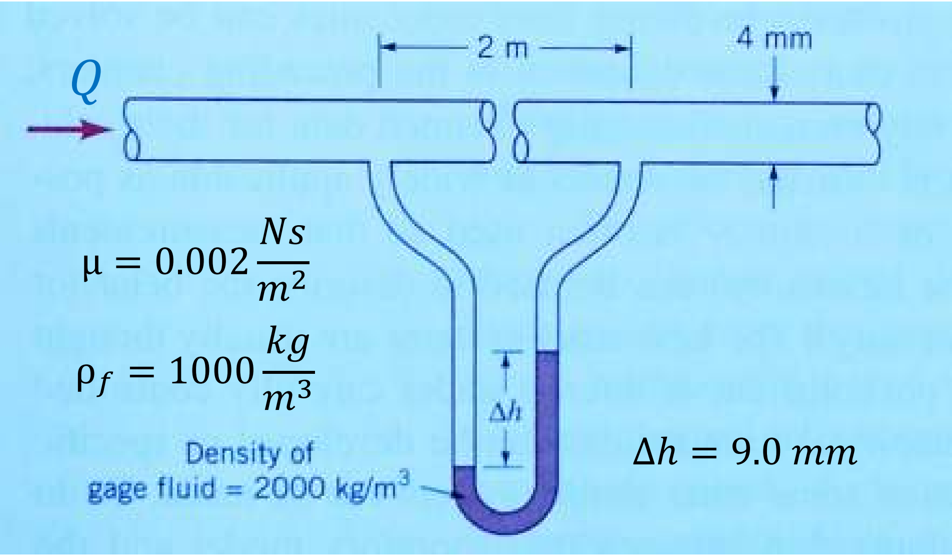
(i) Calculate the flow rate Q in litres per hour assuming laminar flow.

(ii) Use the result of part (i) to check that the flow is laminar.



Example

(i) Poiseuille's Law: $$Q = -\frac{\pi R^4}{8\mu}\frac{\Delta p}{L}$$



- The pressure taps are spaced at $L = 2.0\ m$
- The pressure drop between the pressure taps is:

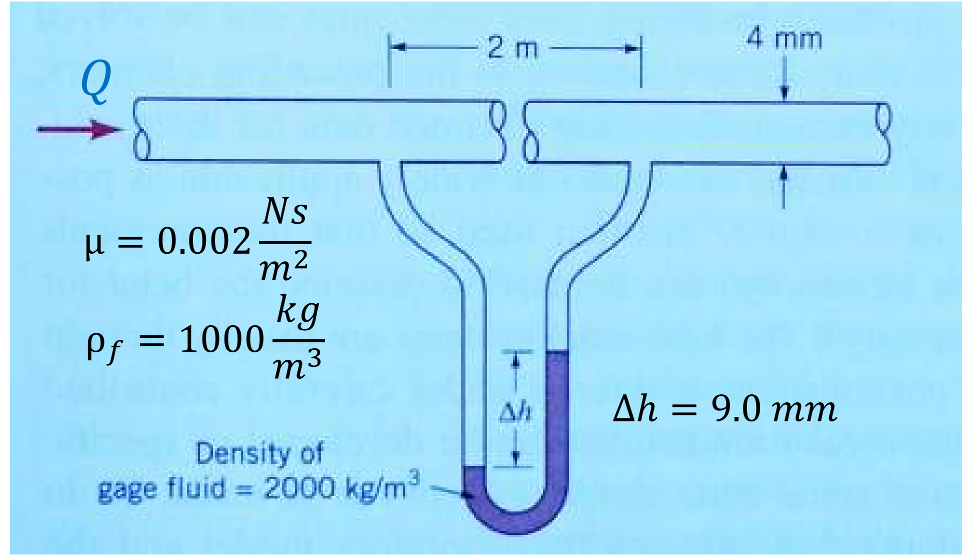
$$\Delta p = -(\gamma_g - \gamma_f)\Delta h = -g(\rho_g - \rho_f)\Delta h = -9.81\frac{m}{s^2}(2000 - 1000)\ \frac{kg}{m^3}\ (0.009m) = -88.29\ \frac{N}{m^2}$$

$$Q = -\frac{\pi(0.002)^4\ m^4}{8\ \left(0.002\frac{Ns}{m^2}\right)}\frac{\left(-88.29\ \frac{N}{m^2}\right)}{2.0\ m} = 1.39\times\ 10^{-7}\frac{m^3}{s}\left(1000\frac{l}{m^3}\right)3600\ \frac{s}{hr} = 0.499\ l/hr$$

Ans.

Example

(ii) Reynolds number: $\mathrm{Re} = \dfrac{\rho \bar{V} D}{\mu}$



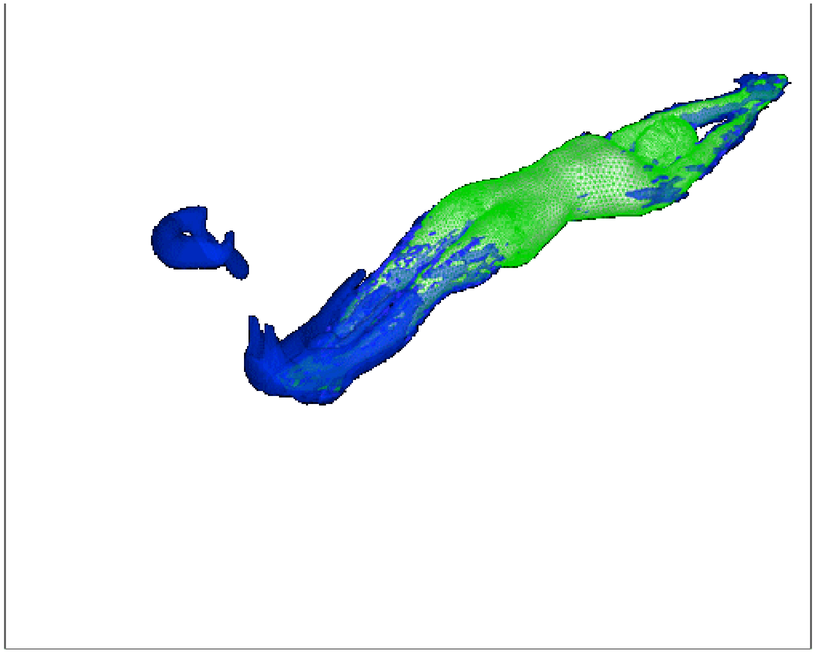
$$Q = \bar{V} A = \bar{V} \frac{\pi D^2}{4}$$

$$\bar{V} = \frac{4Q}{\pi D^2} = \frac{4(1.39 \times 10^{-7} \frac{m^3}{s})}{\pi (0.004)^2 m^2} = 0.0111\, m/s$$

$$\mathrm{Re} = \frac{1000 \frac{kg}{m^3} \left(0.0111 \frac{m}{s}\right) 0.004 m}{0.002 \frac{kg}{s\, m}} = 22.1$$

Ans.

Re is less than ~ 2300. So, the flow is laminar.



Computation Fluid Dynamics Simulation a Swimmer. A 3-D unsteady flow with moving boundaries.
Source: http://fruitsoftheweb.tumblr.com/post/78678686653/female-1

END NOTES

Presentation prepared and delivered by Dr. David Naylor.