MEC516/BME516: Fluid Mechanics I

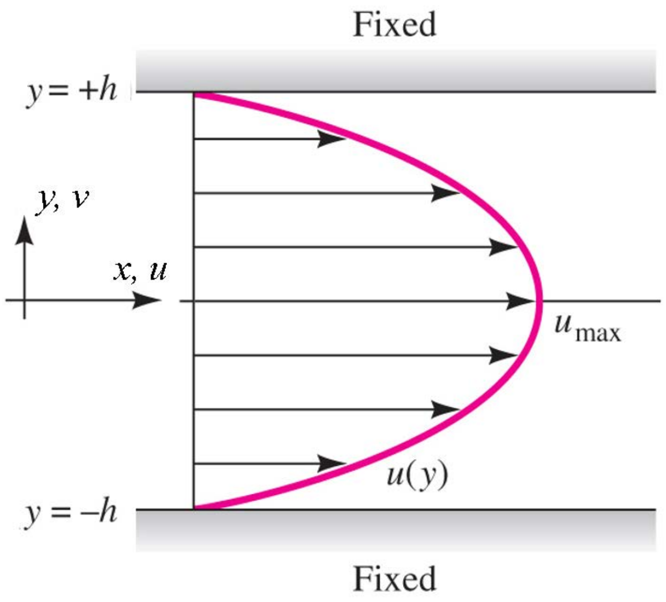
Chapter 4: Differential Relations for Fluid Flow
Part 5

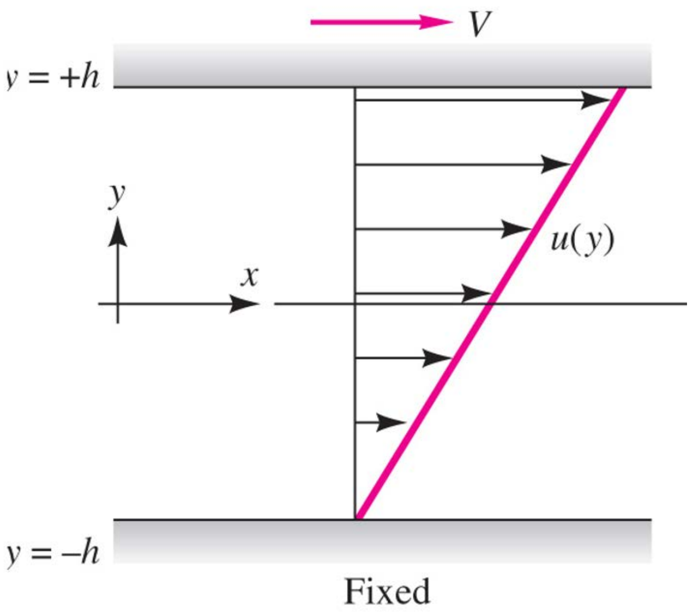
RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering

Overview

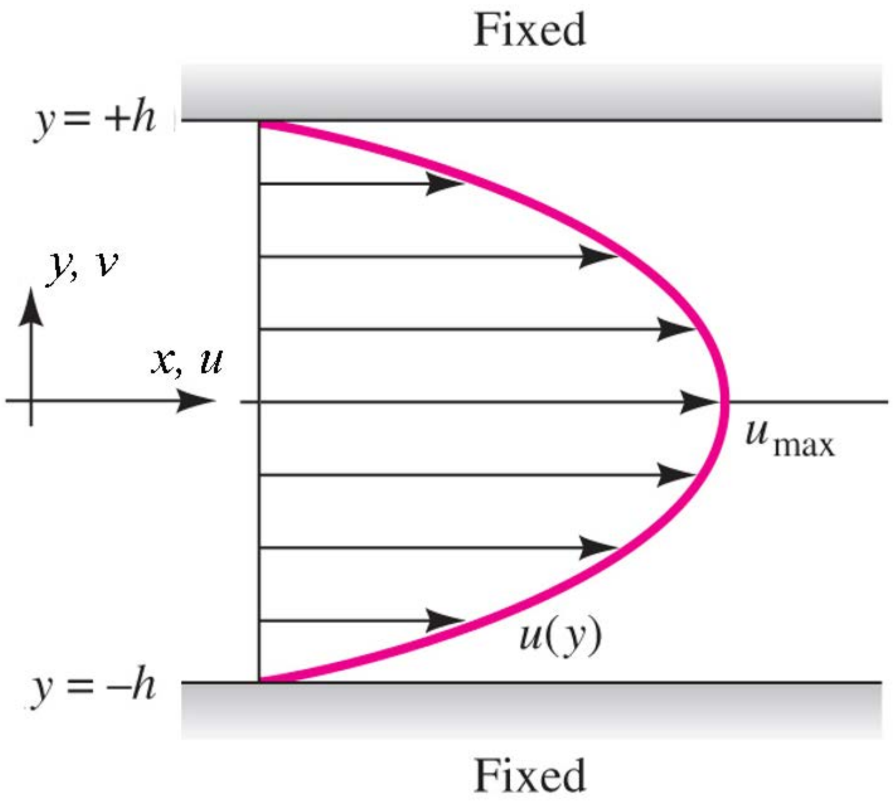
- **Two Exact Solutions to the Continuity and Navier-Stokes Equations**
  - Laminar flow between fixed parallel plates (*Poiseuille Flow).*
  - Laminar flow between parallel plates with one plate moving (*Couette Flow*).

Jean Léonard Marie Poiseuille
(1797-1869)





Laminar Flow Between Fixed Parallel Plates



Problem Definition

- Consider steady laminar incompressible viscous flow between fixed parallel plates a distance $2h$ apart.
- Plates are very wide and long. So, the flow is purely axial: $v = w = 0$.
- Neglect gravity effects. (Adds hydrostatic press. gradient)

Continuity Equation:
(incompressible)

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{v=0} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{w=0} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0$$

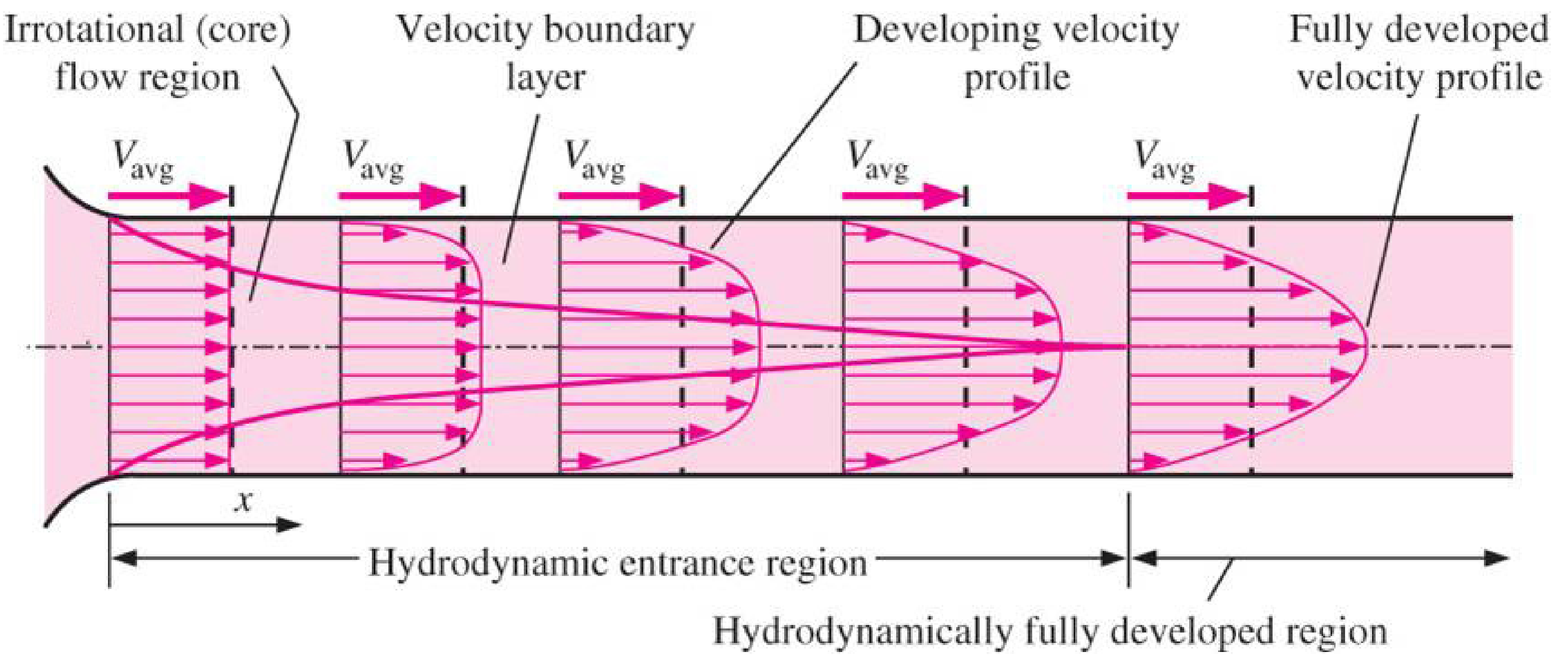
$\therefore u$ is not changing in the x-direction.

- So, the flow is called *fully developed*. No flow entrance effects remain.

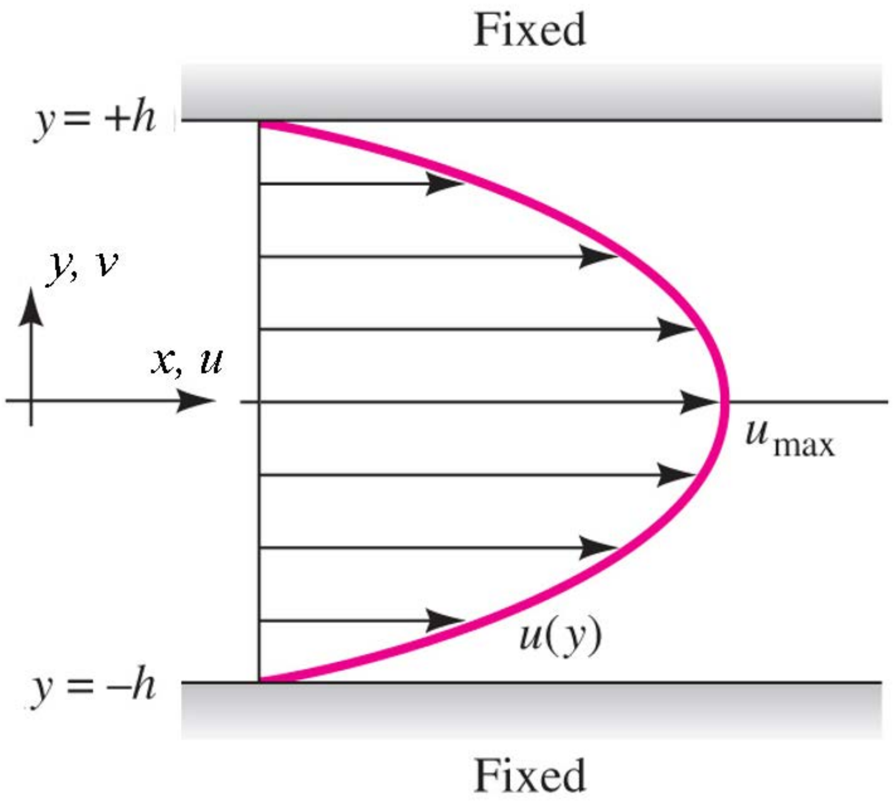
A Brief Side Comment: Developing Flow

- The velocity profile is initially uniform at the inlet.
- *Boundary layers* grow on both walls due to viscous drag.
- Boundary layers merge on the centre line.
- Velocity profile stops changing in x-direction, $\frac{\partial u}{\partial x} = 0$

Current exact solution is for flow that is fully developed, $\frac{\partial u}{\partial x} = 0$



Laminar Flow Between Fixed Parallel Plates



Problem Definition

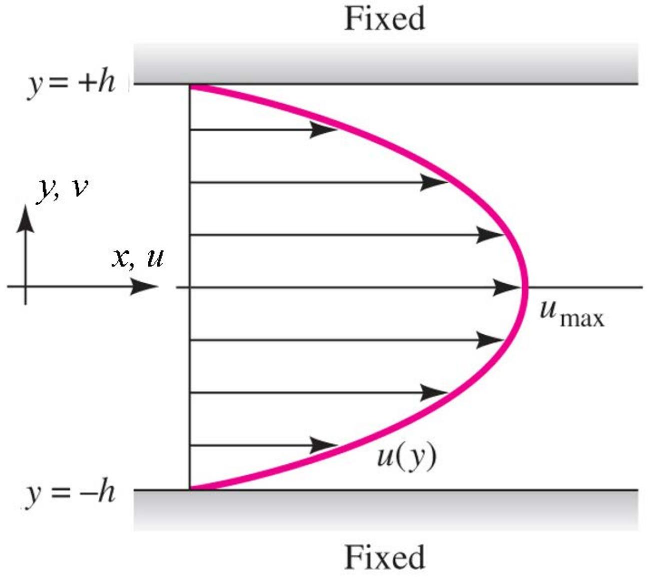
- Consider steady laminar incompressible viscous flow between fixed parallel plates a distance $2h$ apart.
- Plates are very wide and long. So, the flow is purely axial, 1-D flow: $v = w = 0$.
- Neglect gravity effects. (Adds hydrostatic press. gradient in y)

Continuity Equation:
(incompressible)

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{v=0} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{w=0} = 0 \;\rightarrow\; \frac{\partial u}{\partial x} = 0 \quad \therefore u \text{ is not changing in the x-direction.}$$

- So, the flow is called *fully developed*. No flow entrance effects remain.

Laminar Flow Between Fixed Parallel Plates



- We have 1-D flow. So, we write the Navier-Stokes equation in the x-direction.

x-momentum:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \overbrace{\cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}}^{\text{Non-linear convective terms}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) + \cancel{\rho g_x}$$

steady $\quad \frac{\partial u}{\partial x} = 0 \quad v = 0 \quad w = 0 \quad \frac{\partial u}{\partial x} = 0$

- Note that the non-linear terms are zero, which makes the analytical solution much easier.
- Since $v = w = 0$ and gravity is neglected, y- and z- momentum equations give: $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \ p = p(x)$
- Simplifying:

$$\mu\frac{d^2 u}{dy^2} = \frac{dp}{dx} = const$$

Note the full derivatives! This is an Ordinary Differential Equation (ODE) **not** a Partial Differential Equation (PDE).

Laminar Flow Between Fixed Parallel Plates

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = const \underbrace{< 0}_{\text{Press. decreases in x-direction because of viscous shear}}$$

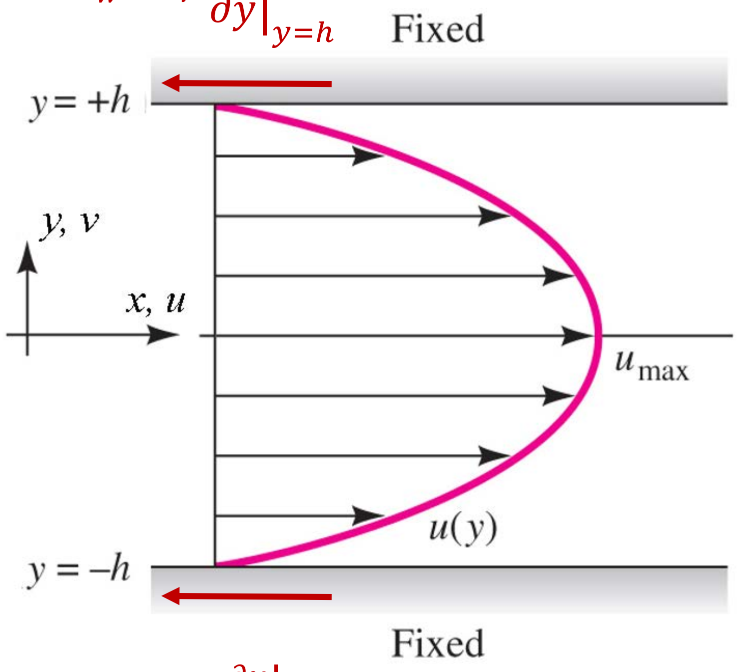
- Why does this expression equal a constant?

- Recall from “separation of variables” method (MTH309): Two equal quantities, one varies with x only $(p(x))$, one varies with y only $(u(y))$. The can be true is only if they equal a constant. Otherwise they would be independent!

- Alternately, we can also use a physical argument to reach the same conclusion:

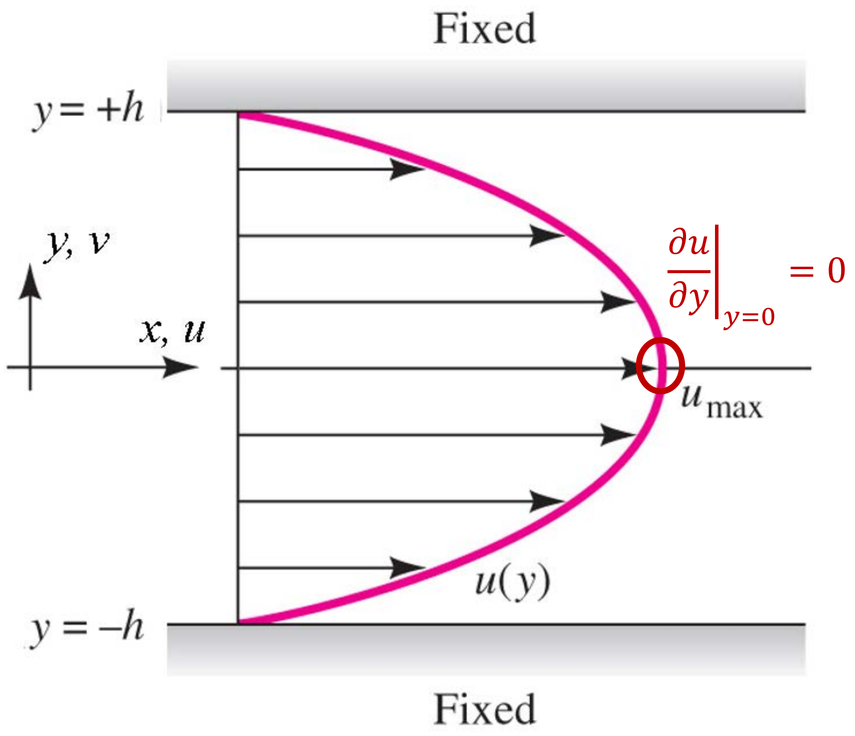
$\frac{dp}{dx}$ is the pressure gradient supplied by the pump to overcome the viscous shear stress at the wall, i.e. the skin friction drag. The shear force at the walls will not change with x since the flow is fully developed, $\frac{\partial u}{\partial x} = 0$. So, the pressure gradient will be constant, i.e. not change with x.

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=h}$$



$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=-h}$$ (force on fluid)

Laminar Flow Between Fixed Parallel Plates



We have: $$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = const$$

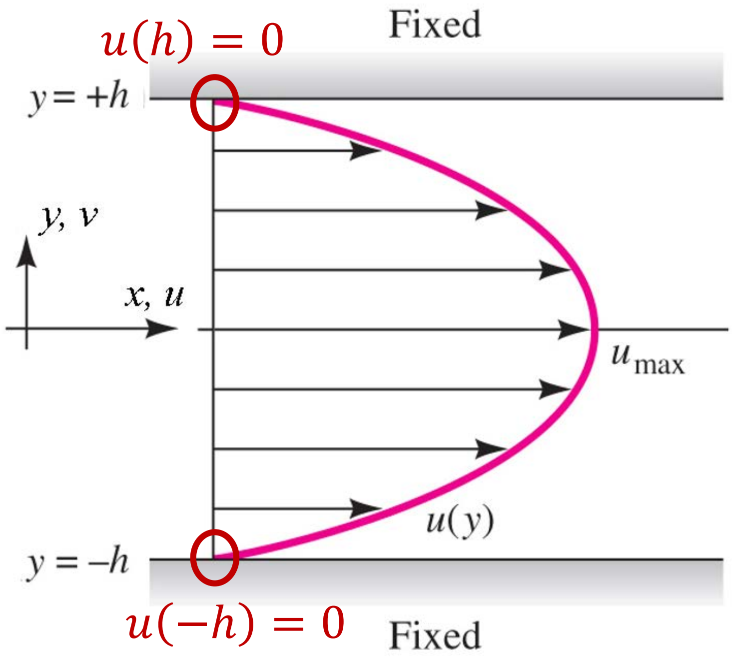
- Integrating with respect to y: $$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

- We can use flow symmetry to evaluate C_1.

The flow is symmetrical about $y = 0$, i.e., $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$ So, $0 = 0 + C_1$

- Gives: C_1=0

Laminar Flow Between Fixed Parallel Plates



$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{dp}{dx}y$$

- Integrating again with respect to y:

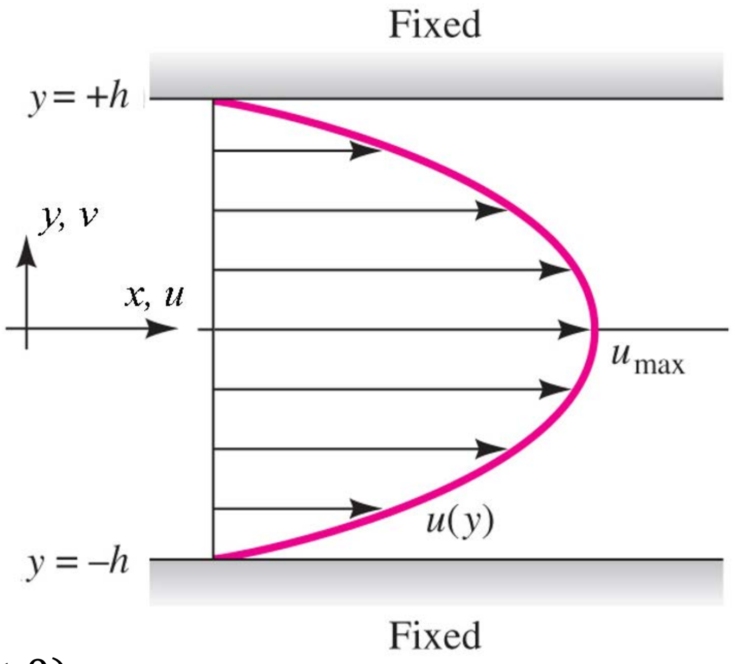
$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_2$$

- We use the boundary conditions at either wall (no-slip) to evaluate and C_2:

At $y = \pm h$ $u = 0$ So, $0 = \frac{1}{2\mu}\frac{dp}{dx}h^2 + C_2$ Thus, $C_2 = -\frac{1}{2\mu}\frac{dp}{dx}h^2$

Laminar Flow Between Fixed Parallel Plates

We have: $u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_2$ and $C_2 = -\frac{1}{2\mu}\frac{dp}{dx}h^2$



- Making the substitution:

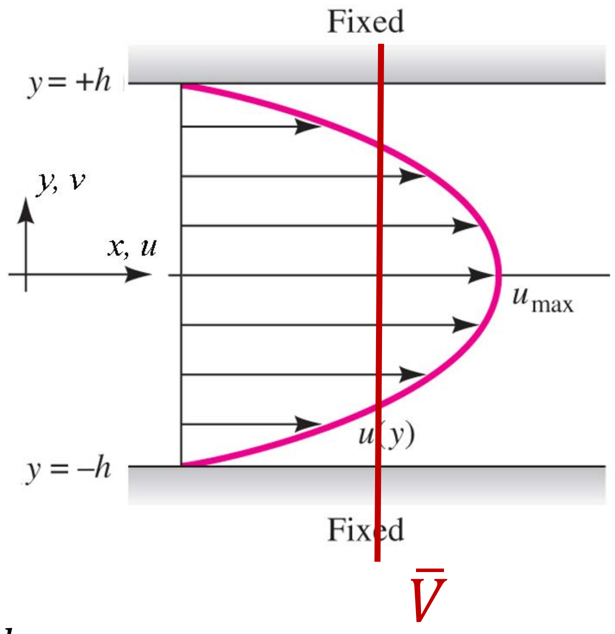
$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right), \text{ So, } u_{max} = -\frac{h^2}{2\mu}\frac{dp}{dx}$$

(Recall $\frac{dp}{dx} < 0$)

$$\boxed{\frac{u}{u_{max}} = \left(1 - \frac{y^2}{h^2}\right)}$$

- The velocity profile for laminar flow between parallel plates is parabolic. Called *Poiseuille flow*, after French physicist (published ~1840).

Laminar Flow Between Fixed Parallel Plates



- We have: $u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right)$

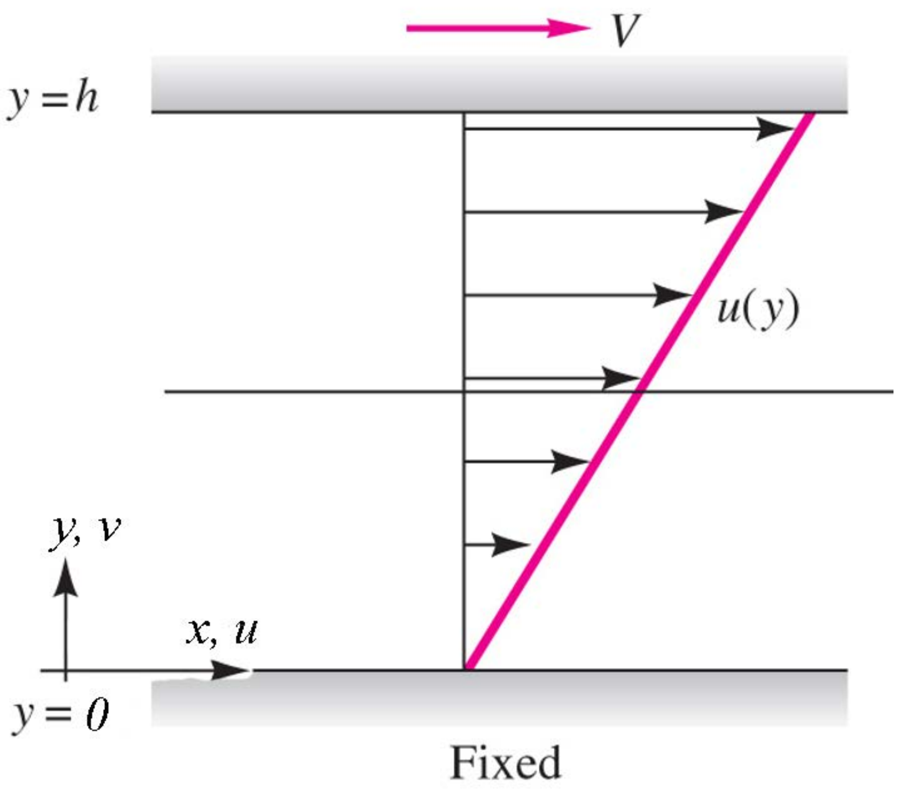
- The volume flow rate is:

$$Q = \int_{-h}^{h} u\, dy = -\frac{h^2}{2\mu}\frac{dp}{dx}\int_{-h}^{h}\left(1 - \frac{y^2}{h^2}\right) dy = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(y - \frac{y^3}{3h^2}\right)\Bigg|_{y=-h}^{y=h} = -\frac{2h^3}{3\mu}\frac{dp}{dx}$$

Noting that: $u_{max} = -\frac{h^2}{2\mu}\frac{dp}{dx}$, we get, $Q = \frac{2}{3}\, u_{max}(2h) = \bar{V}\, 2h$

So, $u_{max} = \frac{3}{2}\,\bar{V}$ (Be careful: This is a different result than for a round pipe.)

Flow Between Fixed Plates with Upper Plate Moving



Problem Definition

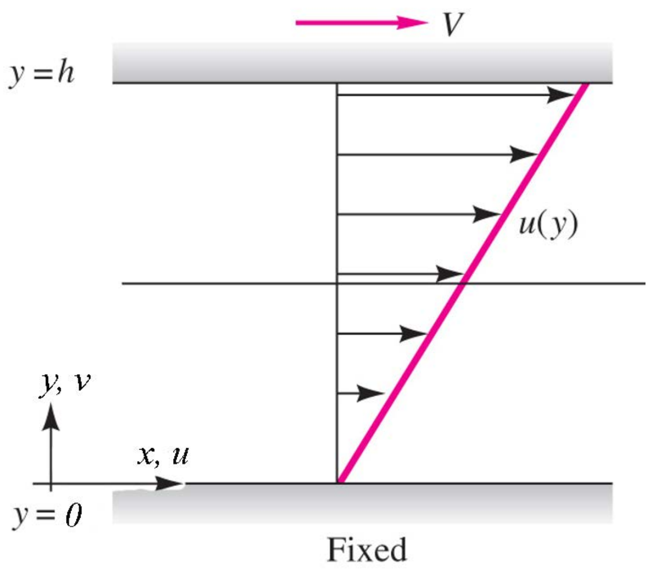
- Consider steady laminar incompressible viscous flow between parallel plates a distance h apart.
- Flow driven by upper plate moving at velocity V . No pressure gradient, $\frac{\partial p}{\partial x} = 0$.
- Plates are very wide and long. So, the flow is purely axial: $v = w = 0$. Neglect gravity effects.

Continuity Equation:
(incompressible)

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{v=0} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{w=0} = 0 \rightarrow \frac{\partial u}{\partial x} = 0$$

So, the flow is fully developed.

Flow Between Fixed Plates with Upper Plate Moving



- So, we have 1-D flow. So, we write the Navier-Stokes equation in the x-direction.

x-momentum:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right) + \cancel{\rho g_x}$$

steady $\quad \frac{\partial u}{\partial x} = 0 \quad v = 0 \quad w = 0 \quad 0 \quad \frac{\partial u}{\partial x} = 0$

- Again, the non-linear terms are zero.
- Simplifying:

$$\frac{d^2 u}{dy^2} = 0$$

- Again, this is now an ODE, not a PDE, since $u = u(y)$

Flow Between Fixed Plates with Upper Plate Moving

- So, we have $$\frac{d^2u}{dy^2} = 0$$

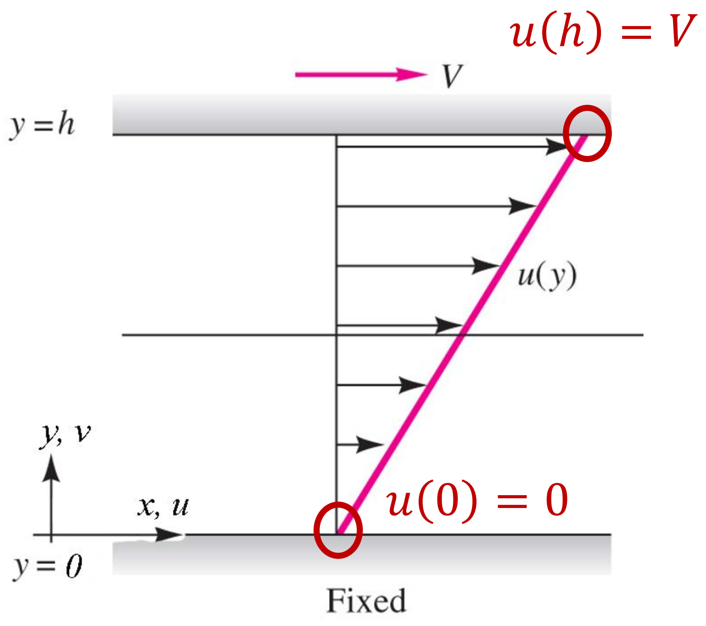
- Integrating twice: $$\frac{du}{dy} = C_1, \quad u = C_1 y + C_2$$

- We use the boundary conditions (B.C.) at $y = 0$ and $y = h$ to get the constants.

- Applying the lower B.C.: $u(0) = 0, \quad 0 = C_1(0) + C_2$ Thus, $C_2 = 0$

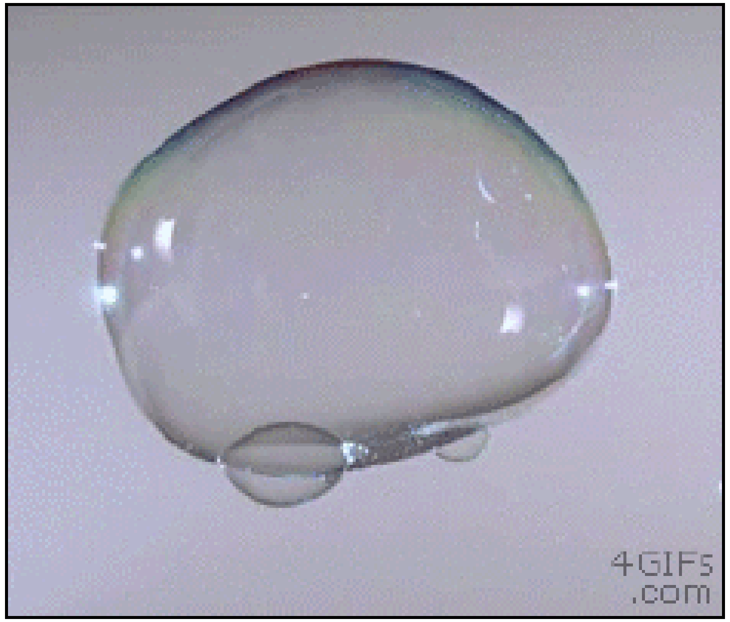
- Applying the upper B.C.: $u(h) = V \quad V = C_1 h$ Thus, $C_1 = \frac{V}{h}$

- The result is: $$\boxed{u = V\,\frac{y}{h}}$$



- The velocity profile is linear. (We made this assumption without proof in Chapter 1)

- Laminar flow between plates with flow driven by the motion of the upper plate is linear is called *Couette flow*, after another French physicist (1858-1943).



Source: http://prguitarman.tumblr.com/post/52221706891/slow-motion-bubble-pop

END NOTES

Presentation prepared and delivered by Dr. David Naylor.