# MEC516/BME516: Fluid Mechanics I

# Chapter 4: Differential Relations for Fluid Flow
# Part 4



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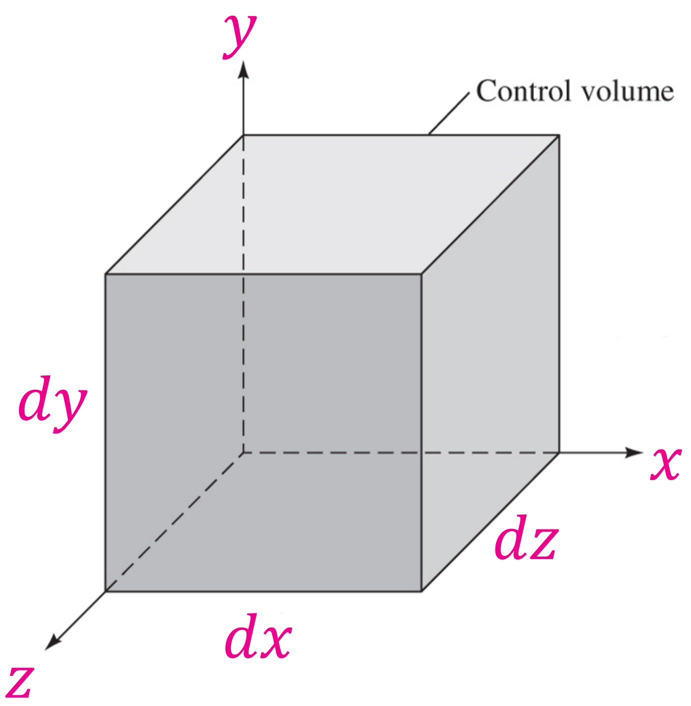
# Overview

- **Differential form of conservation of linear momentum for fluids**
  - Examine the forces on a differential fluid element
  - $\boldsymbol{F} = m\boldsymbol{a}$ for a differential fluid element
  - Navier-Stokes Equations (for a Newtonian fluid)

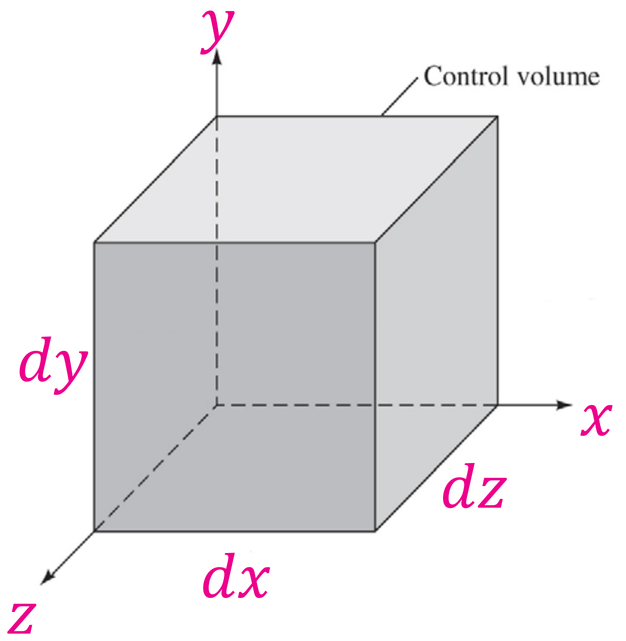
  (This is an overview -- a detailed derivation is beyond the scope of this course)




- **Example**

  Determine if a given velocity vector field $\boldsymbol{V}(x, y, z, t)$ satisfies conservation of momentum.

# Conservation of Linear Momentum




- Consider a differential fluid control volume. The elemental volume has mass $dm = \rho d\forall = \rho\, dx\, dy\, dz$

- Now we apply Newton's second law to this differential mass:

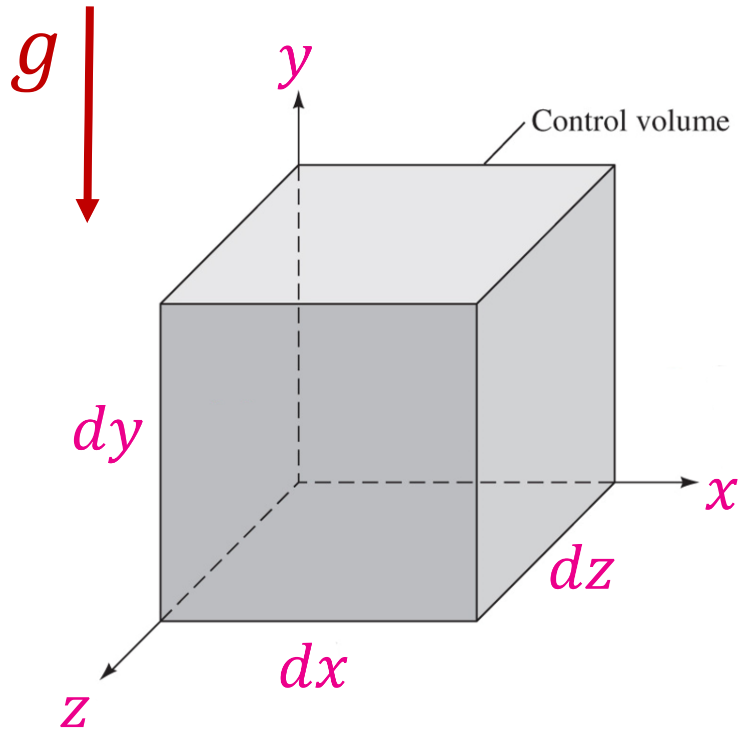
$$\sum \boldsymbol{F} = dm\, \boldsymbol{a} = (\rho\, dx\, dy\, dz)\boldsymbol{a}$$

- We showed that the total fluid acceleration is:

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \frac{D\boldsymbol{V}}{Dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \left(u\frac{\partial \boldsymbol{V}}{\partial x} + v\frac{\partial \boldsymbol{V}}{\partial y} + w\frac{\partial \boldsymbol{V}}{\partial z}\right)$$

- So, we get: $\sum \boldsymbol{F} = \rho \frac{d\boldsymbol{V}}{dt} dx\, dy\, dz = \rho \left\{\frac{\partial \boldsymbol{V}}{\partial t} + \left(u\frac{\partial \boldsymbol{V}}{\partial x} + v\frac{\partial \boldsymbol{V}}{\partial y} + w\frac{\partial \boldsymbol{V}}{\partial z}\right)\right\} dx\, dy\, dz$ Eq. (1)

# Forces on the Differential Fluid Element




- There are two types of forces on the differential fluid element:
  - (i) Body Forces
  - (ii) Surface Forces

<u>(i) Body Forces</u>

- Body forces are uniformly distributed through the element.
- For the purpose of this course, the only body force we will consider is gravity. (Others are possible, e.g. magnetic forces, MHD)

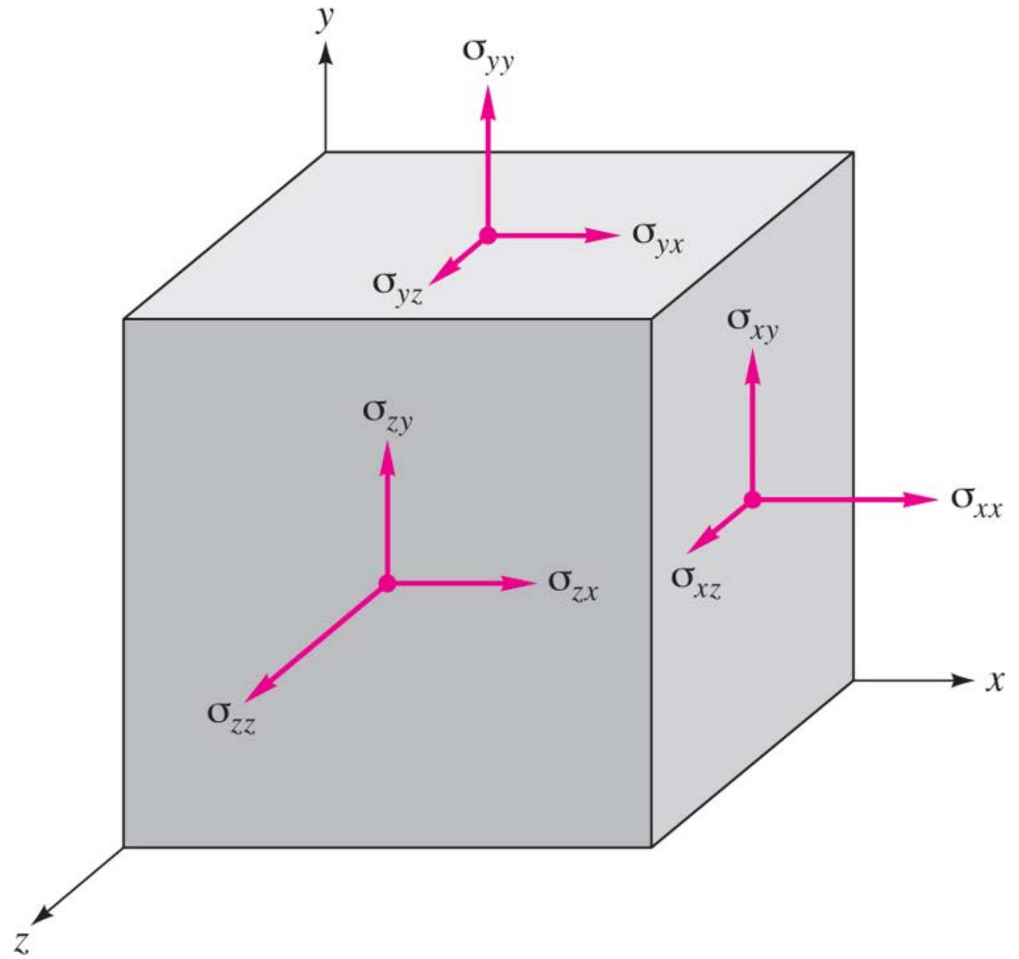
- Define the gravity vector: $\boldsymbol{g} = g_x\mathbf{i} + g_y\mathbf{j} + g_z\boldsymbol{k}$

The gravity force on the element is $\boldsymbol{dF_{grav}} = \rho\, \boldsymbol{g}\, dx\, dy\, dz$

Often, the y-axis is upward, then: $g_x = 0,\; g_y = -g,\; g_z = 0$

# Forces on the Differential Fluid Element

$\sigma_{ij}$ = Stress in $j$ direction on a face normal to $i$ axis




<u>(ii) Surface Forces</u>

- The surface forces per unit area (stresses) on the fluid are due to:
  - viscous stress ($\tau$)
  - pressure ($p$)

- This gives rise to nine components of stress:

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

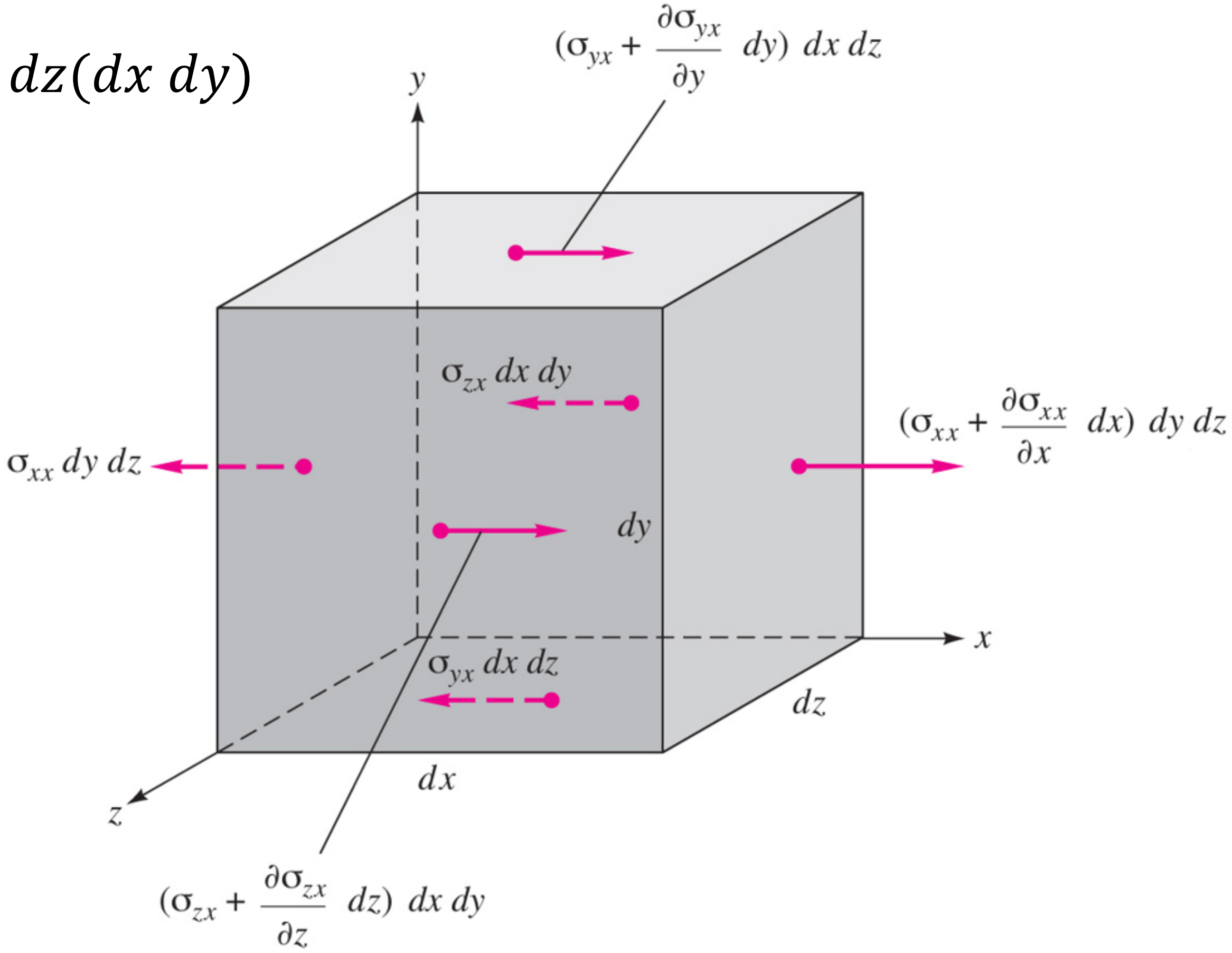
← x-direction
← y-direction
← z-direction

# Forces on the Differential Fluid Element

- The net surface forces in the x-direction:

$$dF_x = \frac{\partial \sigma_{xx}}{\partial x} dx(dydz) + \frac{\partial \sigma_{yx}}{\partial y} dy\,(dxdz) + \frac{\partial \sigma_{zx}}{\partial z} dz(dx\,dy)$$

$$dF_x = \{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\}\, dx\, dy\, dz$$




Similarly,

- The net surface forces in the y-direction:

$$dF_y = \{\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\}\, dx\, dy\, dz$$

- The net surface forces in the z-direction:

$$dF_z = \{\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\}\, dx\, dy\, dz$$

# Forces on the Differential Fluid Element

- For a Newtonian fluid, it can be shown that the viscous stresses are related to the velocity field as follows:

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

These are the "drag" forces on the fluid caused by fluid viscosity, i.e. fluid friction.

- Now we apply Newton's second law, Eq. (1):

$$\sum \boldsymbol{F} = \rho \frac{d\boldsymbol{V}}{dt} dx\, dy\, dz = \rho \left\{\frac{\partial \boldsymbol{V}}{\partial t} + \left(u \frac{\partial \boldsymbol{V}}{\partial x} + v \frac{\partial \boldsymbol{V}}{\partial y} + w \frac{\partial \boldsymbol{V}}{\partial z}\right)\right\} dx\, dy\, dz \qquad \text{Eq. (1)}$$

As discussed, the sum of the forces $\sum \boldsymbol{F}$ include: (i) pressure forces, (ii) viscous forces, and (iii) gravity force.

# The Navier-Stokes Equations

For a Newtonian fluid with constant properties ($\rho = const, \mu = const$):

x-momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x$$

y-momentum:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y$$

z-momentum:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z$$

# The Navier-Stokes Equations

These equation represents conservation of momentum in the 3 coordinate directions. They are **F**=m**a** on a per unit volume basis.

Let's look at the terms in the x-equation, and the units:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

| $\rho$ | $\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$ | | $-\frac{\partial p}{\partial x}$ | $\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$ | $\rho g_x$ |
|---|---|---|---|---|---|
| mass per unit volume | acceleration (local and convective) | | pressure force per unit vol. | viscous force per unit vol. | gravity force per unit vol. |
| $\frac{kg}{m^3}$ | $\frac{m}{s^2}$ | $=$ | $\frac{N}{m^3}$ | $\frac{N}{m^3}$ | $\frac{N}{m^3}$ |

# Navier-Stokes Equations

- The constant property Navier-Stokes equations can be written in a compact form using vector notation:

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}\right)=-\nabla p+\ \mu\,\nabla^2\boldsymbol{V}+\rho\boldsymbol{g}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplace Operator.

# Cylindrical Coordinates

- As previously discussed, it is sometimes desirable to work in cylindrical coordinates.
- The constant property Navier-Stokes equations are given in Appendix D:

The $r$-momentum equation:

$$\frac{\partial v_r}{\partial t} + (\mathbf{V} \cdot \nabla) v_r - \frac{1}{r} v_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) \qquad \text{(D.5)}$$

The $\theta$-momentum equation:

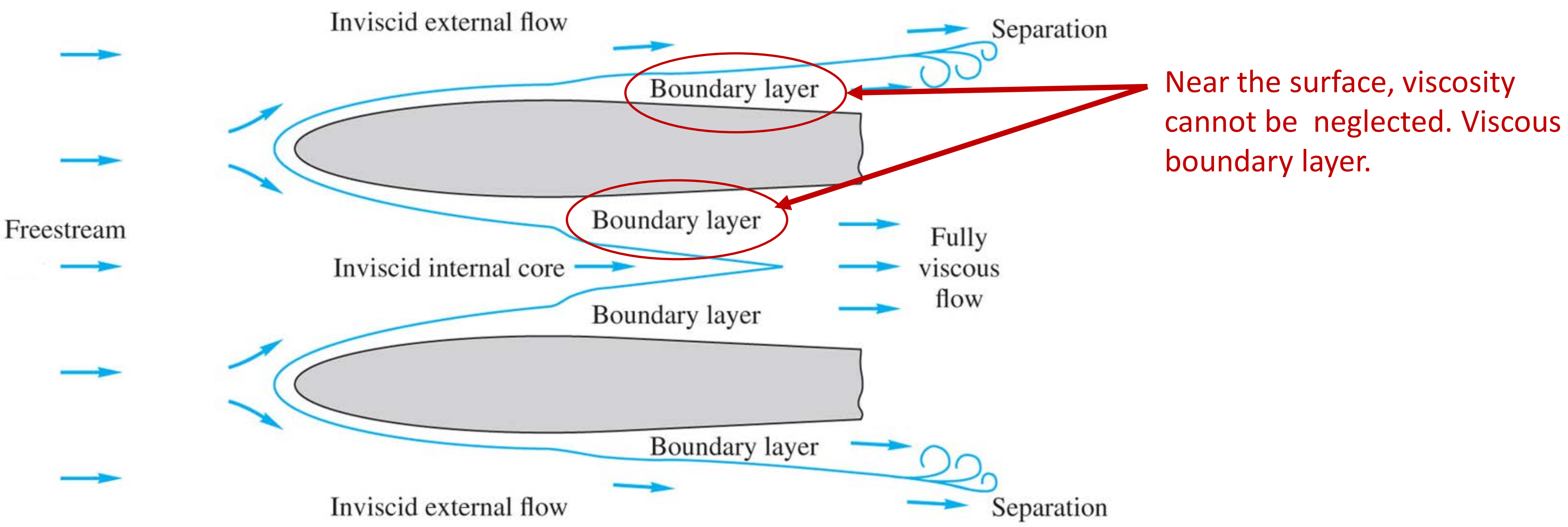
$$\frac{\partial v_\theta}{\partial t} + (\mathbf{V} \cdot \nabla) v_\theta + \frac{1}{r} v_r v_\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) \qquad \text{(D.6)}$$

The $z$-momentum equation:

$$\frac{\partial v_z}{\partial t} + (\mathbf{V} \cdot \nabla) v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu \nabla^2 v_z \qquad \text{(D.7)}$$

# Inviscid Flow and Euler's Equation

- Flows with no viscosity are called *Inviscid Flows*. All real fluids have viscosity → idealization.
- Far away from solid surfaces, real flows can be *approximated* as inviscid. Flow fields behaves like μ=0.




- For inviscid flows ($\mu = \tau_{ij} = 0$) the Navier-Stokes equations simplify to:

$$\rho \frac{d\boldsymbol{V}}{dt} = -\nabla p + \rho \boldsymbol{g}$$

This is called *Euler's Equation.*

# Inviscid Flow and Euler's Equation

$$\rho \frac{d\boldsymbol{V}}{dt} = -\nabla p + \rho \boldsymbol{g}$$

*Euler's Equation* can be integrated along a streamline to give the Bernoulli Equation.

**History**

- This equation was derived (in 1757) ***before*** the equations for viscous flow.
- Leonhard Euler, $18^{th}$ century Swiss engineer, physicist and mathematician.
    - Classical beam theory, Euler buckling load
    - Euler formula: $e^{ix} = \cos(x) + i\,\sin(x)$

Leonhard Euler, 1707-1783

# Example

Consider steady incompressible flow of a Newtonian fluid with the velocity field:

$$\boldsymbol{V} = -2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j} + 0\,\boldsymbol{k}$$

Neglect gravitational effects. Does this velocity field satisfy the conservation of momentum i.e., the Navier-Stokes equations?

<u>Solution</u>

This is a two-dimensional flow. We need to consider the conservation of momentum in the x- and y-directions.

# Example

$$\boldsymbol{V} = -2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j} + 0\,\boldsymbol{k}$$

$$u = -2xy \quad v = y^2 - x^2 \quad w = 0$$

x-momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x$$

Making the substitutions:

$$\rho(0 + (-2xy)(-2y) + (y^2 - x^2)(-2x) + 0) = -\frac{\partial p}{\partial x} + \mu\,(0 + 0 + 0) + 0$$

$$\frac{\partial p}{\partial x} = \rho(-4xy^2 + 2xy^2 - 2x^3) = -2\rho(xy^2 + x^3)$$

# Example

$$u = -2xy \quad v = y^2 - x^2 \quad w = 0$$

y-momentum:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y$$

Making the substitutions:

$$\rho(0 + (-2xy)(-2x) + (y^2 - x^2)(2y) + 0) = -\frac{\partial p}{\partial y} + \mu\,(-2 + 2 + 0) + 0$$

$$\frac{\partial p}{\partial y} = \rho(-4yx^2 - 2y^3 + 2yx^2) = -2\rho(yx^2 + y^3)$$

# Example

- So far we have shown that: $\frac{\partial p}{\partial x} = -2\rho(xy^2 + x^3) \quad and \quad \frac{\partial p}{\partial y} = -2\rho(yx^2 + y^3)$

- The pressure field $p(x, y)$ is a single function. To determine if the gradients in the x and y directions are compatible, we evaluate the mixed derivative $\frac{\partial^2 p}{\partial x \partial y}$

- For a single function, these two mixed derivatives must be the equal:

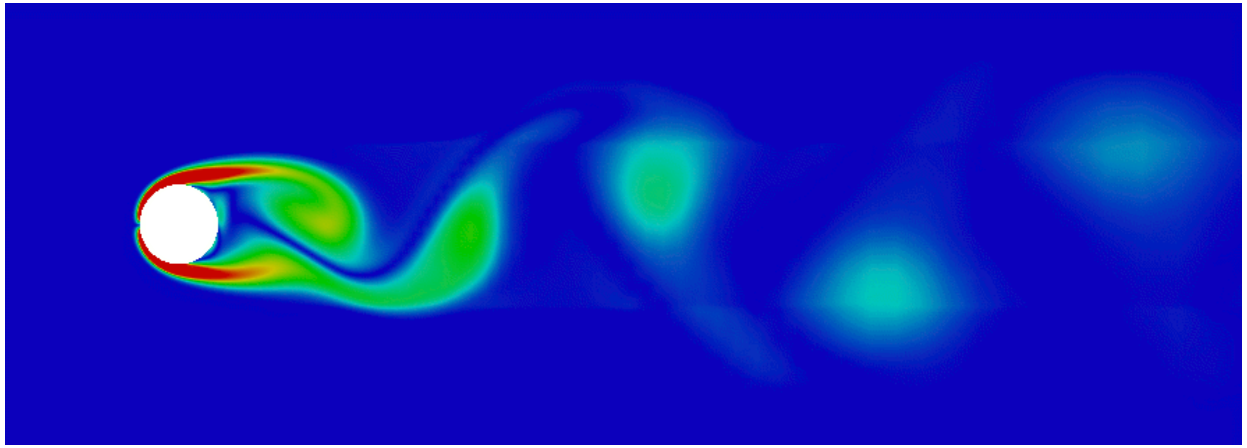
$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial y}\left(-2\rho\left(xy^2 + x^3\right)\right) = -4\rho xy$$

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2\rho(yx^2 + y^3)\right) = -4\rho xy$$

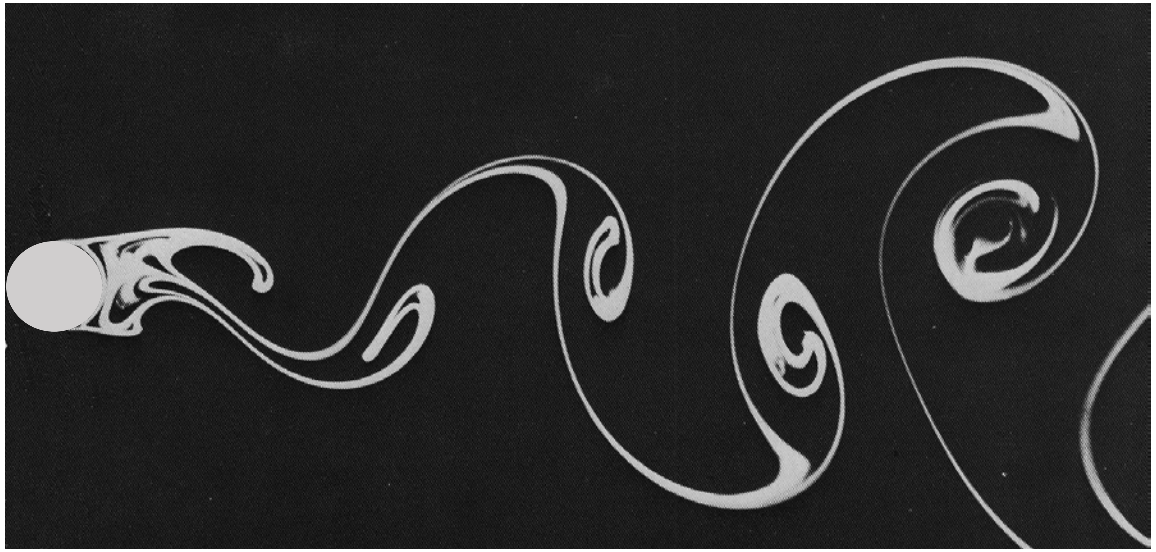
Thus, the velocity field satisfies the Navier-Stokes equations.

Vortex shedding from a cylinder in cross flow Re≈150. Solution of the unsteady 2D Navier-Stokes equations.

CFD Prediction



Experimental Flow Visualization



## END NOTES

Presentation prepared and delivered by Dr. David Naylor.



Sources:
www.redditian.com
Album of Fluid Motion