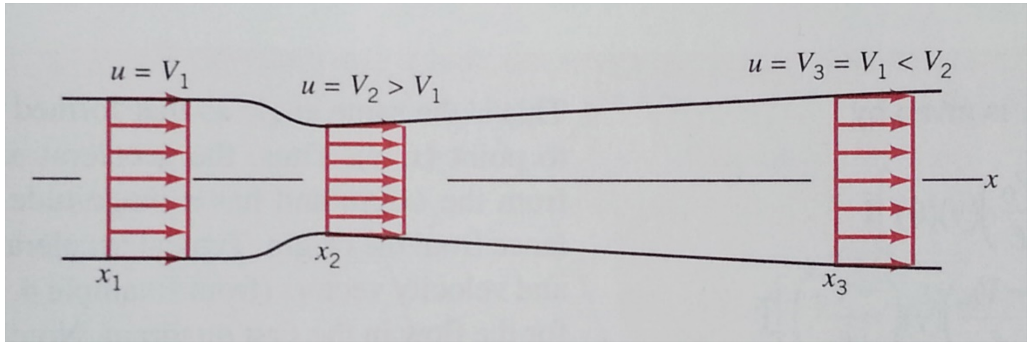
MEC516/BME516: Fluid Mechanics I

Chapter 4: Differential Relations for Fluid Flow Part 3

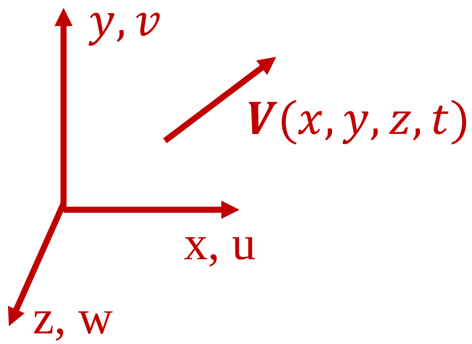
RYERSON UNIVERSITY
Department of Mechanical & Industrial Engineering

Overview

- **Derivation of the differential expression for the fluid acceleration field**
  - "Local" versus "convective" acceleration
  - Vector notation



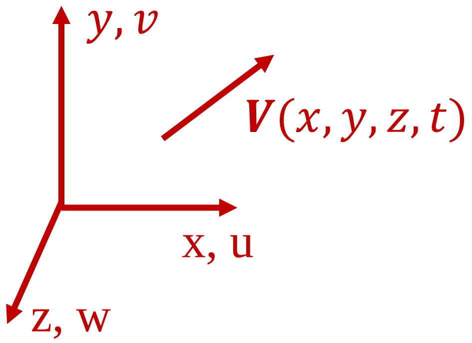
- **Examples**
  i) Calculating fluid acceleration in a steady flow in a nozzle (convective acceleration).
  ii) Calculating fluid acceleration field given a velocity vector field, $\boldsymbol{V}(x, y, z, t)$



This is vital background for an upcoming discussion of conservation of linear momentum (**F**=m**a** for fluids) in differential form i.e., for the *Navier-Stokes* equations.

The Fluid Acceleration Field

- We have been describing fluid flow with a Eulerian velocity vector field:



$$\boldsymbol{V} = u(x, y, z, t)\mathbf{i} + v(x, y, z, t)\mathbf{j} + w(x, y, z, t)\mathbf{k}$$

- Thus, the acceleration field is:

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \frac{du}{dt}\mathbf{i} + \frac{dv}{dt}\mathbf{j} + \frac{dw}{dt}\mathbf{k}$$

- Note that each velocity component is a function of four variables, e.g. $u = u(x, y, z, t)$.
- So, considering only the x-component, to get the total acceleration we must use the chain rule:

$$\frac{du\,(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

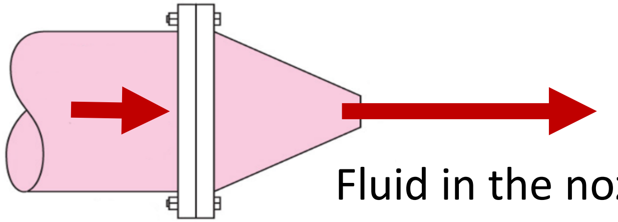
The Fluid Acceleration Field

$$\frac{du\,(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

- Note that: $\frac{\partial x}{\partial t} = u \quad \frac{\partial y}{\partial t} = v \quad \frac{\partial z}{\partial t} = w$
- Making the substitution, we get:

$$\frac{du\,(x,y,z,t)}{dt} = \underbrace{\frac{\partial u}{\partial t}}_{\text{local acceleration}} + \overbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}^{\text{convective acceleration}}$$

- *Local acceleration* is caused by local unsteady flow e.g., pump speeding up, Q ↑ in time.
- *Convective acceleration* is caused by fluid moving into a region with different velocity. e.g. steady flow through a nozzle.



Fluid in the nozzle accelerates in a steady flow!

The Fluid Acceleration Field

- So the acceleration vector field $\boldsymbol{a}$ is:

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \frac{D\boldsymbol{V}}{Dt} = \underbrace{\frac{\partial \boldsymbol{V}}{\partial t}}_{\text{local acceleration}} + \overbrace{\left(u\frac{\partial \boldsymbol{V}}{\partial x} + v\frac{\partial \boldsymbol{V}}{\partial y} + w\frac{\partial \boldsymbol{V}}{\partial z}\right)}^{\text{convective acceleration}}$$

Not an ordinary derivative!

- Most books use the notation $\dfrac{D\boldsymbol{V}}{Dt}$ to distinguish this from an ordinary derivative (variable that is a function of only one variable). (The Frank White textbook does not.)

- $\dfrac{d\boldsymbol{V}}{dt}$ or $\dfrac{D\boldsymbol{V}}{Dt}$ is called the *material derivative* or the *substantial derivative*.

The Fluid Acceleration Field

$$u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$$

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \frac{D\boldsymbol{V}}{Dt} = \frac{\partial \boldsymbol{V}}{\partial t} + \left(u\frac{\partial \boldsymbol{V}}{\partial x} + v\frac{\partial \boldsymbol{V}}{\partial y} + w\frac{\partial \boldsymbol{V}}{\partial z} \right)$$

- The three scalar components can be written as:

x-direction:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

y-direction:

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

z-direction:

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

The Fluid Acceleration Field

- Vector Notation. This is approaching a “graduate level” discussion...

$$u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$$

$$\boldsymbol{a} = \frac{\partial \boldsymbol{V}}{\partial t} + \left(u\frac{\partial \boldsymbol{V}}{\partial x} + v\frac{\partial \boldsymbol{V}}{\partial y} + w\frac{\partial \boldsymbol{V}}{\partial z} \right) = \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}}_{\text{vector notation (compact)}}$$

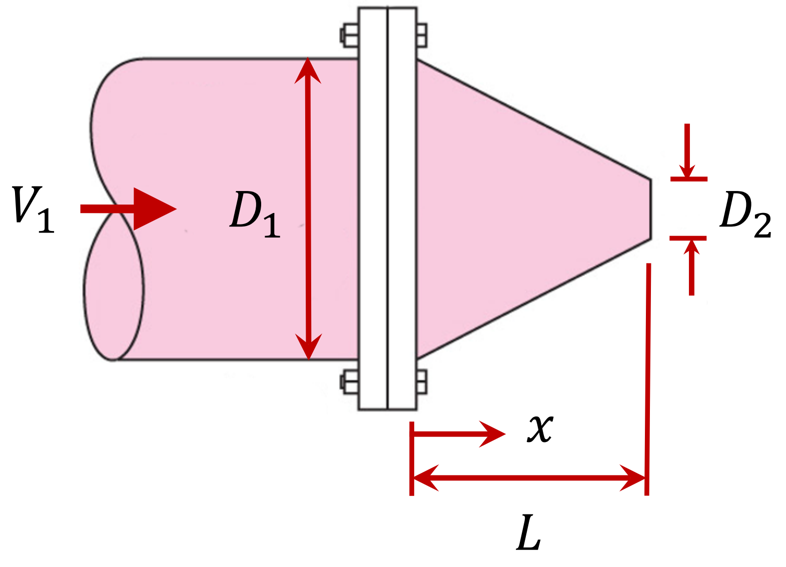
where symbol "∇" is the *vector gradient* (or *del*) operator: $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$

- So, $\boldsymbol{V}\cdot\nabla$ becomes an operator: $\boldsymbol{V}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

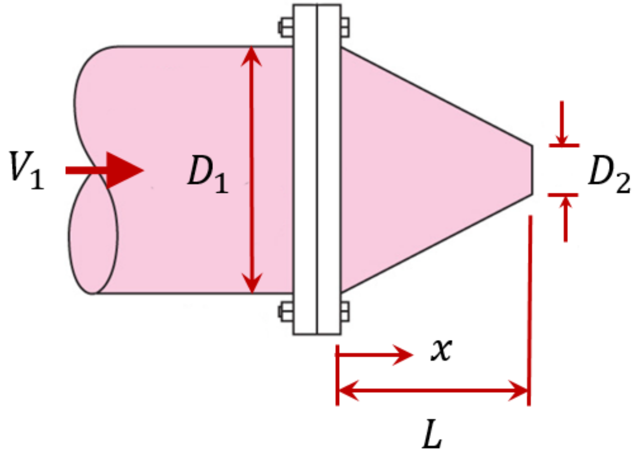
Example 1

Consider a **steady** one-dimensional flow of liquid in nozzle. The incompressible fluid enters the nozzle at velocity V_1. The nozzle has length L, and inlet diameter D_1 and outlet diameter D_2.

Derive an analytical expression of the fluid acceleration in the nozzle as a function of the x-coordinate, measured from the nozzle inlet.



Example 1



- Start with general vector expression for fluid acceleration:

$$\boldsymbol{a} = \underbrace{\cancel{\frac{\partial \boldsymbol{V}}{\partial t}}}_{0 \text{ (steady)}} + \left(u\frac{\partial \boldsymbol{V}}{\partial x} + \underbrace{\cancel{v\frac{\partial \boldsymbol{V}}{\partial y}}}_{0 \text{ (1-D)}} + \underbrace{\cancel{w\frac{\partial \boldsymbol{V}}{\partial z}}}_{0 \text{ (1-D)}}\right)$$

- Flow is steady. Pump flow rate is not changing. Thus: $\frac{\partial \boldsymbol{V}}{\partial t} = 0$

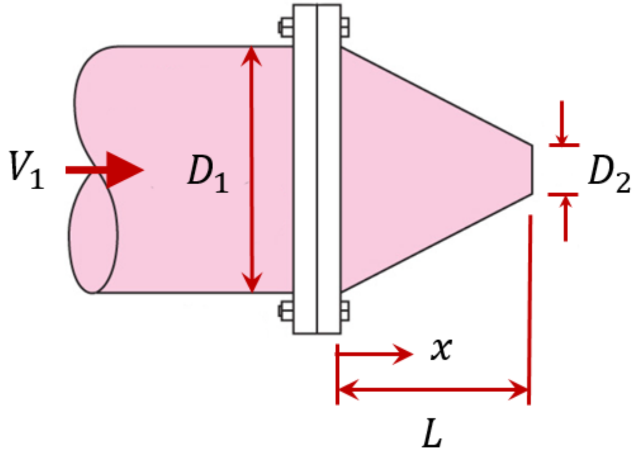
- Flow is 1-D: $v = w = 0$ Thus: $\boldsymbol{V} = u\,\mathbf{i}$

- So, the acceleration equation becomes: $a_x = u\frac{du}{dx}$ (ordinary derivative, $a_x = a_x(x)$ only)

- Next, we need an expression for u(x).

Example 1

$$\boxed{a_x = u\frac{du}{dx}}$$



- Fluid is incompressible: $Q = VA = const \rightarrow \quad V_1 A_1 = u(x)A(x)$

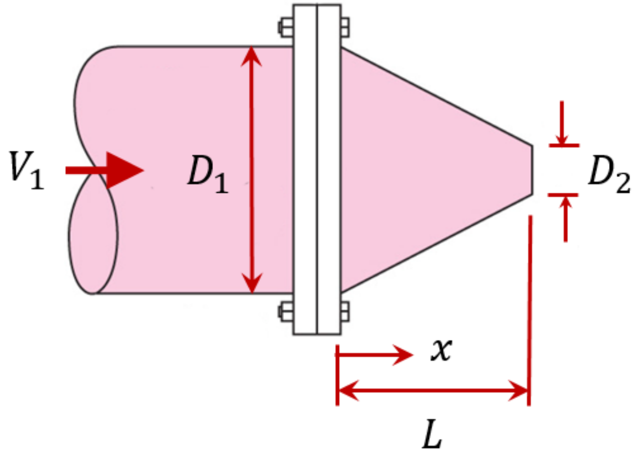
- From continuity, we note that: $\quad u(x) = \left(\frac{A_1}{A(x)}\right)V_1 = \left(\frac{D_1}{D(x)}\right)^2 V_1$

- The nozzle diameter varies linearly with x: $D(x) = D_1 - \left(\frac{D_1 - D_2}{L}\right)x$

- Making the substitution: $u(x) = \frac{D_1^2\, V_1}{\left(D_1 - \left(\frac{D_1 - D_2}{L}\right)x\right)^2}$

- So, the acceleration in x-direction becomes: $a_x = u\frac{du}{dx} = \left[\frac{D_1^2\, V_1}{\left(D_1 - \left(\frac{D_1 - D_2}{L}\right)x\right)^2}\right]\frac{-2\, D_1^2\, (-(D_1 - D_2)/L)\, V_1}{\left(D_1 - \left(\frac{D_1 - D_2}{L}\right)x\right)^3}$

Example 1



- We have the x-acceleration: $a_x = \left[\frac{D_1^2\, V_1}{\left(D_1 - \left(\frac{D_1 - D_2}{L}\right)x\right)^2}\right] \frac{2\, D_1^2\, (D_1 - D_2)\, V_1}{L\left(D_1 - \left(\frac{D_1 - D_2}{L}\right)x\right)^3}$

- Combining terms:

$$a_x = \frac{2\, D_1^4\, (D_1 - D_2) V_1^2}{L\left(D_1 - \left(\frac{D_1 - D_2}{L}\right)x\right)^5}$$

Ans.

- Check units: $\frac{m^4 m \frac{m^2}{s^2}}{m\, m^5} = \frac{m}{s^2}$

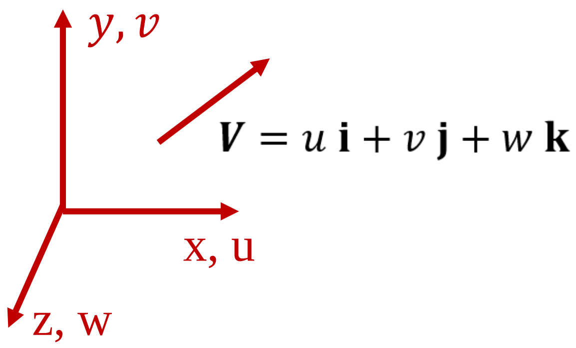
Comments:

- a_x is positive throughout the nozzle. Min. at nozzle inlet, $x = D_1$ & max. at nozzle exit, $x = D_2$.
- This acceleration is purely convective, since the flow is <u>steady</u>. The acceleration occurs because the flow is convected into a region of higher velocity.
- The fluid acceleration produces a thrust on the nozzle in the negative x-direction, $F_x = \dot{m}\,(V_2 - V_1)$

Example 2

Consider a hypothetical fluid velocity vector field given by:

$$\boldsymbol{V} = Ct(x^2 - y^2)\,\mathbf{i} \;-2Cxyt\,\mathbf{j} + 3y\,\mathbf{k}$$



(i) Is the flow field steady or unsteady?

(ii) Obtain an expression for the acceleration vector $\boldsymbol{a}$.

(iii) Evaluate the acceleration vector $\boldsymbol{a}$ at $(x, y, z, t) = (1, 1, 1, 1)$

Solution

(i) The flow is unsteady (or transient) since time (t) appears in the velocity vector (which will give rise to local acceleration.)

Example 2

(ii) We have: $$\boldsymbol{V} = \overbrace{C\,t(x^2 - y^2)}^{u}\,\mathbf{i} \; - \overbrace{2Cxyt}^{v}\,\mathbf{j} + \overbrace{3y}^{w}\,\mathbf{k}$$

- The x-component of acceleration is: $$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

- Evaluating terms: $$a_x = C(x^2 - y^2) + Ct(x^2 - y^2)\,(2Cxt) + (-2Cxyt)(-2Cyt) + 3y(0)$$

$$a_x = C(x^2 - y^2) + 2\,C^2\,x\,t^2(x^2 - y^2) + 4C^2x\,y^2\,t^2$$

Example 2

We have:

$$\boldsymbol{V} = \underbrace{C\,t(x^2 - y^2)}_{u}\,\mathbf{i} \underbrace{- 2Cxyt}_{v}\,\mathbf{j} + \underbrace{3y}_{w}\,\mathbf{k}$$

- The y-component:

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

- Evaluating terms:

$$a_y = -2Cxy + Ct(x^2 - y^2)\,(-2Cyt) + (-2Cxyt)(-2Cxt) + 3y(0)$$

$$a_y = -2Cxy\; - 2C^2\,yt^2(x^2 - y^2) + 4\,C^2x^2yt^2$$

Example 2

We have:

$$\boldsymbol{V} = \overbrace{C\,t(x^2 - y^2)}^{u}\,\mathbf{i} \;\overbrace{-\,2Cxyt}^{v}\,\mathbf{j} + \overbrace{3y}^{w}\,\mathbf{k}$$

- The z-component:

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

- Evaluating terms: $a_z = 0 + Ct(x^2 - y^2)\,(0) + (-2Cxyt)3 + 3y(0),\;\; a_z = -6\,Cxyt$

- Simplifying some of the terms, the acceleration vector is:

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \{C(x^2 - y^2) + 2\,C^2\,x\,t^2(x^2 - y^2) + 4C^2x\,y^2\,t^2\}\,\mathbf{i}$$

$$+\{-2Cxy - 2C^2\,yt^2(x^2 - y^2) + 4\,C^2x^2yt^2\}\,\mathbf{j}\; - \{6Cxyt\}\,\mathbf{k}$$

Example 2

We have:

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \{C(x^2 - y^2) + 2\,C^2\,x\,t^2(x^2 - y^2) + 4C^2 x\,y^2\,t^2\}\,\mathbf{i}$$

$$+\{-2Cxy + \;-2C^2yt^2(x^2 - y^2) + 4\,C^2x^2yt^2\}\,\mathbf{j}\; - \{6Cxyt\}\,\mathbf{k}$$

(iii) Evaluating the acceleration vector $\boldsymbol{a}$ at $(x, y, z, t) = (1, 1, 1, 1)$:

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \{4C^2\}\,\mathbf{i}\; + \;\{-2C + 4\,C^2\}\,\mathbf{j}\; - \{6C\}\,\mathbf{k}$$

Ans.

Source: http://mattybing1025.tumblr.com

END NOTES

Presentation prepared and delivered by Dr. David Naylor.