# MEC516/BME516: Fluid Mechanics I

# Chapter 4: Differential Relations for Fluid Flow
# Part 2



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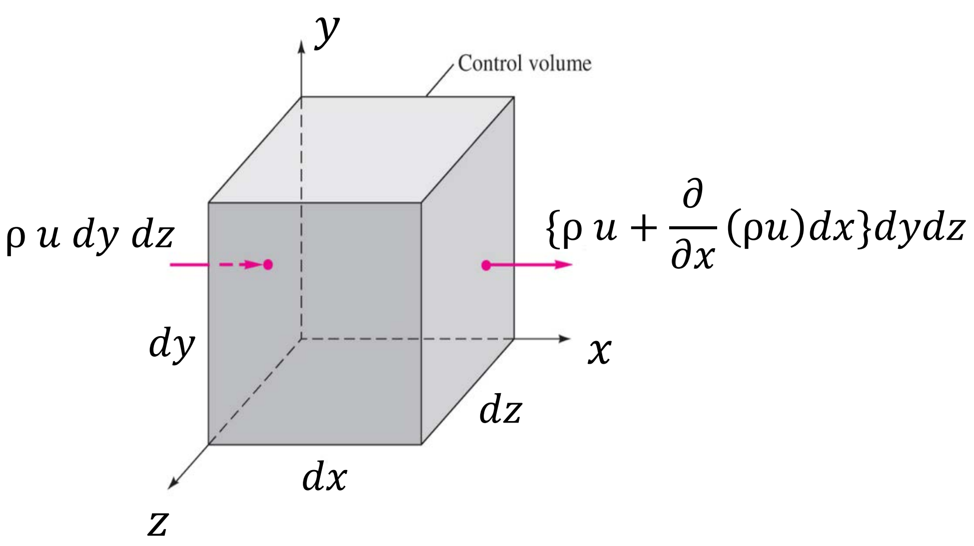
# Overview

- **Derivation of the differential equation for conservation of mass: The *Continuity Equation***
  - Vector notation
  - Simplification for incompressible flow.
  - Cylindrical coordinates

- **Example**
  - Given a velocity vector field, $\boldsymbol{V}(x, y, z, t)$, determine if the continuity equation is satisfied.




$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

# Derivation of the Continuity Equation

- Derivation in Cartesian coordinates
- Consider a differential control volume with dimensions $dx\,dy\,dz$
- Located in a velocity field with arbitrary velocity $\boldsymbol{V}$ (x, y, z, t), and density fields ρ(x, y, z, t)
- Consider x-direction:

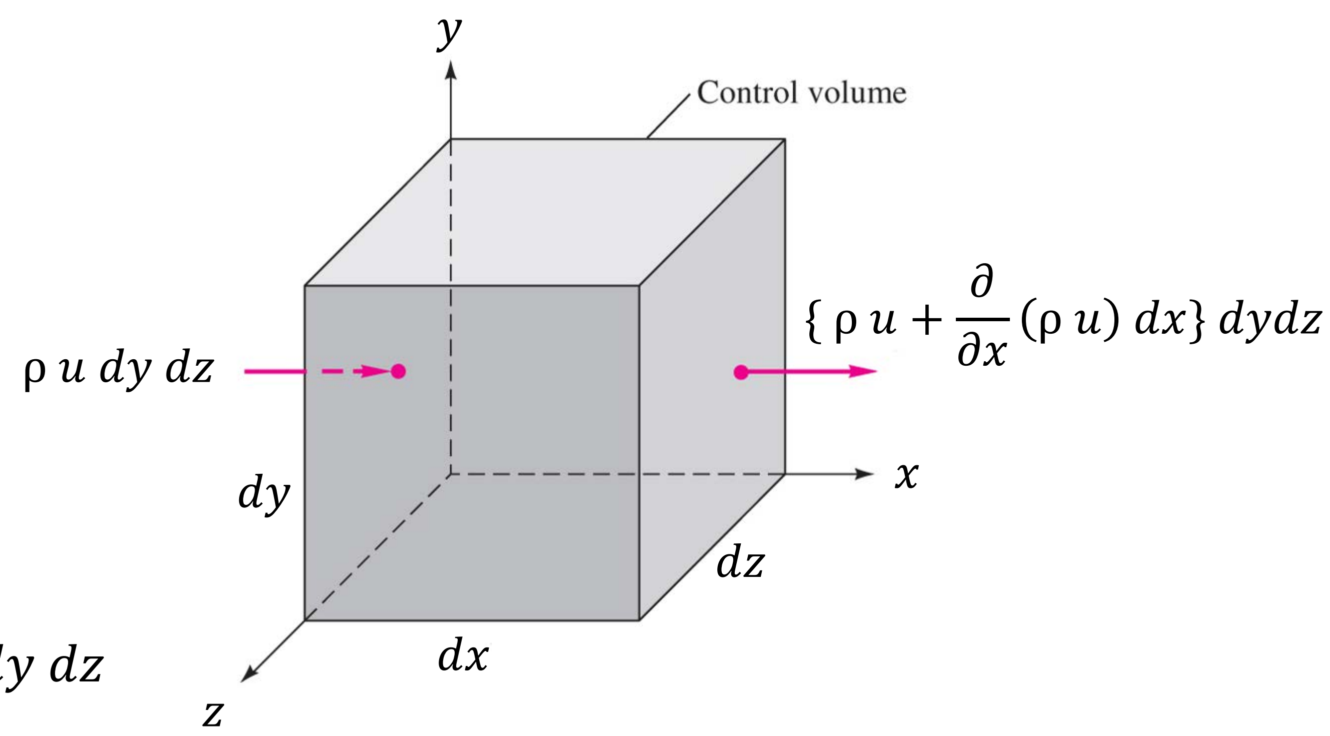
Could be 3D, unsteady & compressible

Mass flow rate into the c.v.: $\rho\, u\, dy\, dz$

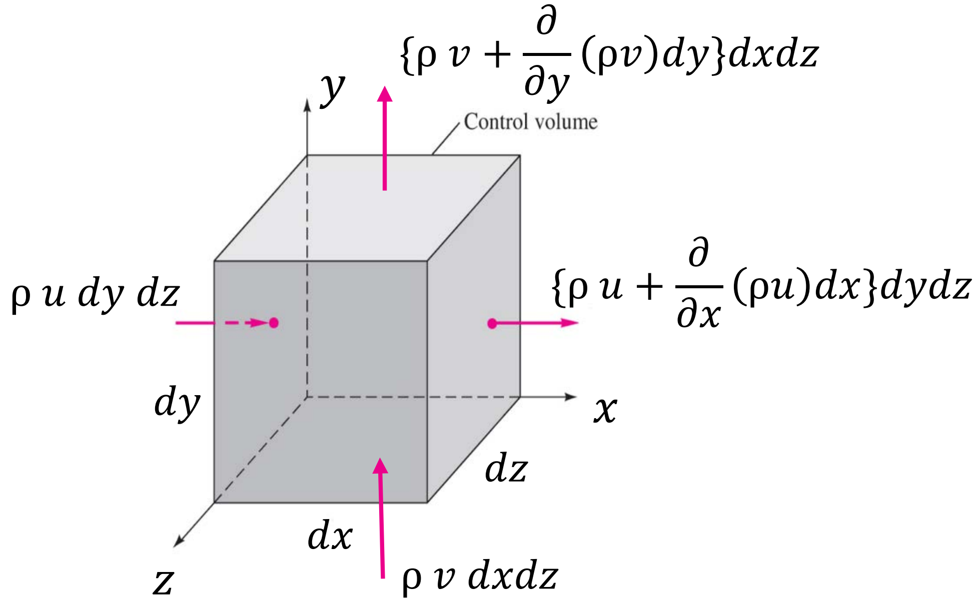
Mass flow rate out of c.v.:

$$\{\rho\, u + \frac{\partial}{\partial x}(\rho\, u)\, dx + \frac{1}{2!}\frac{\partial^2}{\partial x^2}(\rho\, u)\,(dx)^2 + \cdots\}\, dy\, dz$$

Neglect higher order terms.
They vanish as dx→ 0



# Derivation of the Continuity Equation




- Consider the inflow and outflow on the other faces:

| Face | Inlet mass Flow | Outlet mass flow |
|---|---|---|
| $x$ | $\rho u\; dy\; dz$ | $\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz$ |
| $y$ | $\rho v\; dx\; dz$ | $\left[\rho v + \frac{\partial}{\partial y}(\rho v)\, dy\right] dx\, dz$ |
| $z$ | $\rho w\; dx\; dy$ | $\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx\, dy$ |

# Derivation of the Continuity Equation

- The flow through each side can be consider 1-D. Flow at a point as dx, dy, dz →0
- From Chapter 3 (Control Volume Analysis), conservation of mass was expressed as:

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = \sum_{inlets} \rho_{in} A_{in} V_{in} - \sum_{out} \rho_{out} A_{out} V_{out}$$

- This is simple mass "accounting" → Rate of Storage = Rate in – Rate out

- For a differential volume $d\forall = dx\, dy\, dz$, the storage term reduces to:

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = \frac{\partial \rho}{\partial t} d\forall = \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

# Derivation of the Continuity Equation

| Face | Inlet mass flow | Outlet mass flow |
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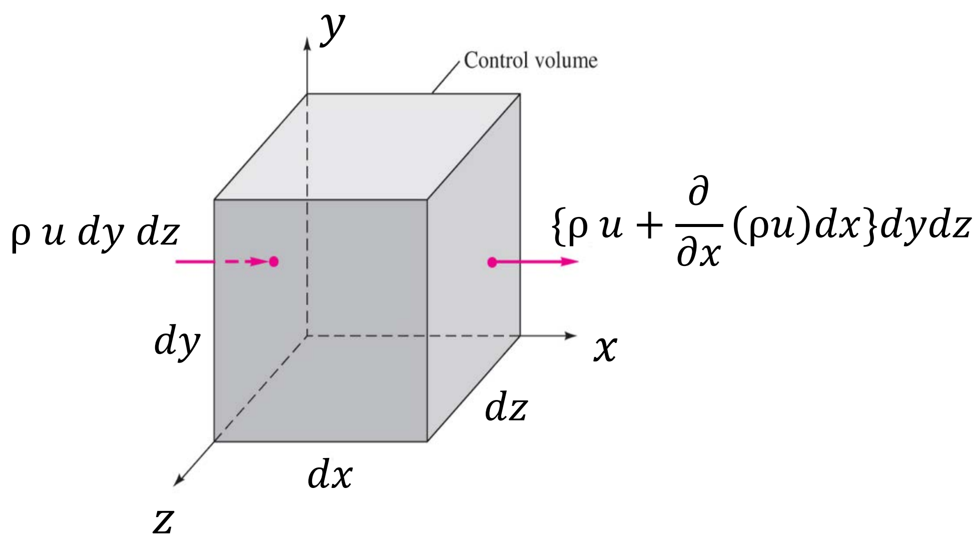



$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz = \sum_{inlets} \rho_{in} A_{in} V_{in} - \sum_{out} \rho_{out} A_{out} V_{out}$$

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz = -\left\{\frac{\partial}{\partial x}(\rho u) dx\right\} dydz - \left\{\frac{\partial}{\partial y}(\rho v) dy\right\} dx\, dz - \left\{\frac{\partial}{\partial z}(\rho\, w)\, dz\right\} dx\, dy$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

General *Continuity Equation*

# Derivation of the Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

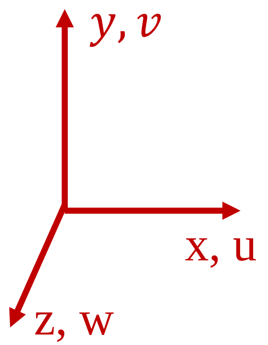
General *Continuity Equation*

Valid for (i) steady/unsteady flow, (ii) viscous or frictionless flow, (iii) compressible/incompressible flow.

Vector notation, in terms of the velocity vector $\boldsymbol{V}$:

Note: This is a dot product

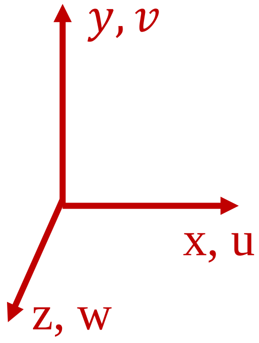
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$




where symbol "$\nabla$" is the *vector gradient* (or *del*) operator:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

# Derivation of the Continuity Equation




$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

- For incompressible flow $\rho = const$. So, $\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0.$
  Continuity becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

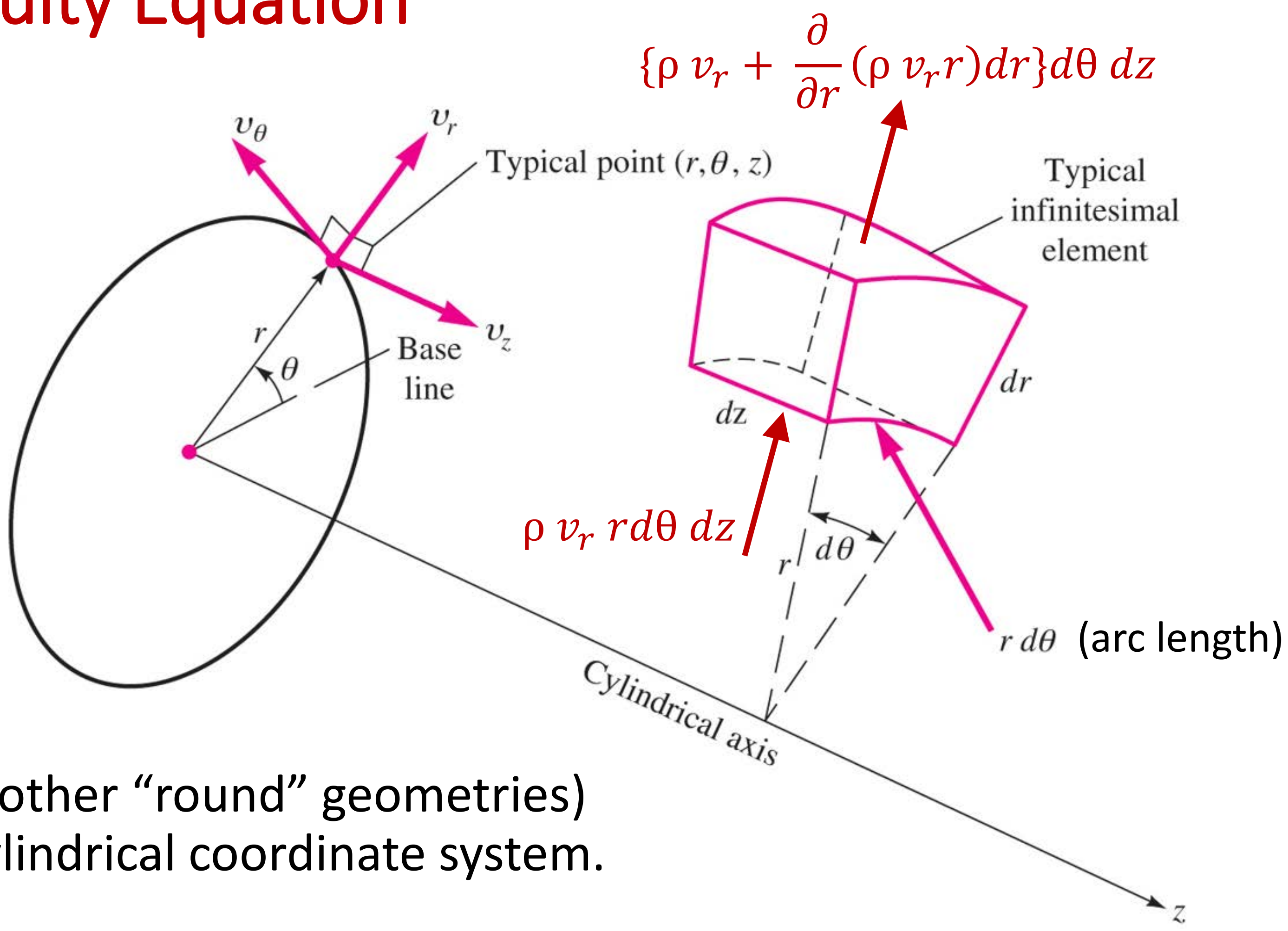
(Mach Number $Ma = \frac{V}{c} \leq \sim 0.3$)

Valid for <u>steady or unsteady</u> incompressible flow. In vector notation:

$$\nabla \cdot \boldsymbol{V} = 0$$

- The scalar product of the del operator and a vector is called the *Divergence*.
- For incompressible flows the divergence of the velocity vector field is zero.

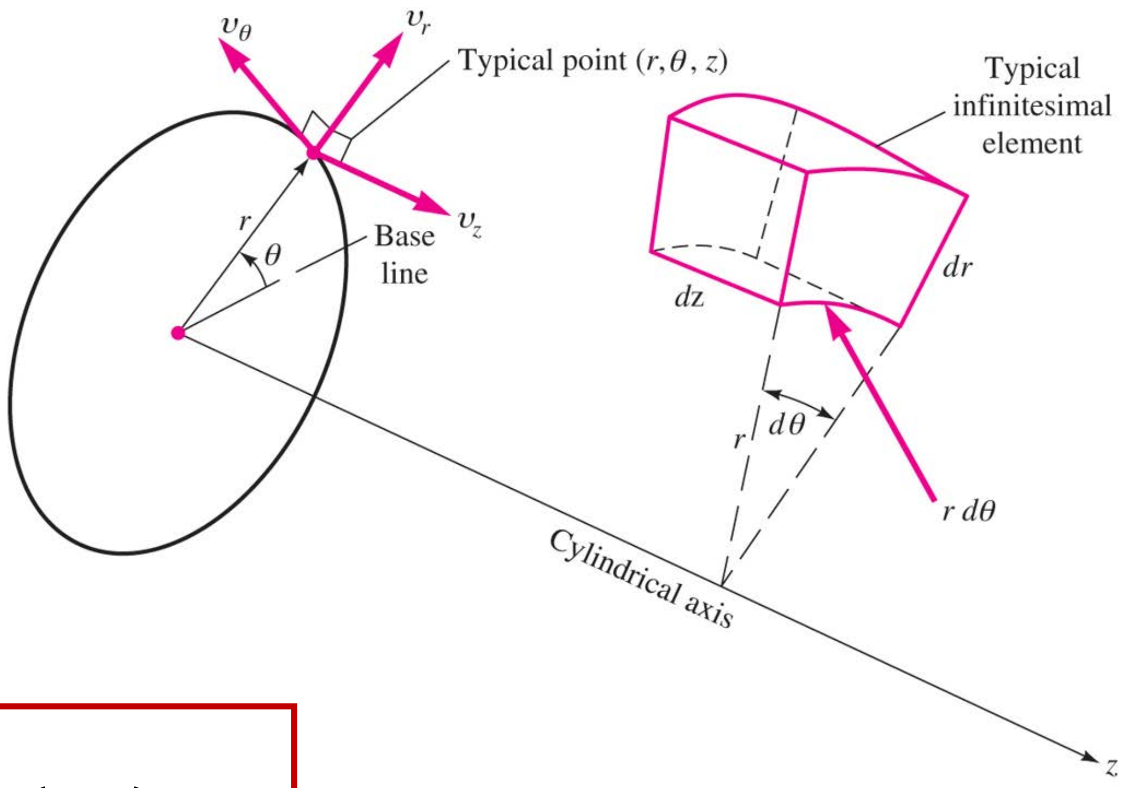
# Derivation of the Continuity Equation




- For analyzing flow in pipes (and other "round" geometries) it is often convenient to use a cylindrical coordinate system.
- Velocity field: $V(r, z, \theta, t)$

$$\boldsymbol{V} = v_r\,(r, z, \theta, t)\mathbf{i} + v_z(r, z, \theta, t)\mathbf{j} + v_\theta(r, z, \theta, t)\mathbf{k}$$

# Derivation of the Continuity Equation




- The Continuity equation in cylindrical coordinates:

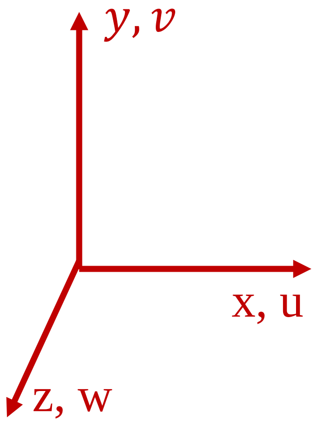
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$where\ \boldsymbol{V} = v_r(r, z, \theta, t)\mathbf{i} + v_z(r, z, \theta, t)\mathbf{j} + v_\theta(r, z, \theta, t)\mathbf{k}$$

# Example

An engineer claims to have found a solution for a particular incompressible flow. In Cartesian coordinates, the proposed solution is:

$$\boldsymbol{V} = \underbrace{(4x + 2y + 3z)}_{u}\,\mathbf{i} + \underbrace{(2x - 3y + 3z)}_{v}\,\mathbf{j} + \underbrace{(3x + 2y + 2z)}_{w}\,\mathbf{k}$$




Determine:

(a) Is the flow steady or unsteady?

(b) Does this velocity field satisfy conservation of mass?

Ans. (a): **Flow is steady**, because time does not appear in any of the velocity components.

$$u \neq u(t),\ \ v \neq v(t),\ \ w \neq w(t)$$

# Example

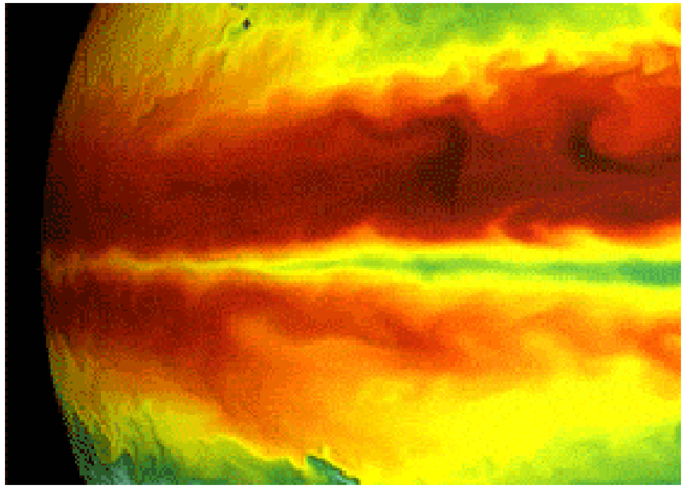
We have: $\boldsymbol{V} = \underbrace{(4x + 2y + 3z)}_{u}\,\mathbf{i} + \underbrace{(2x - 3y + 3z)}_{v}\,\mathbf{j} + \underbrace{(3x + 2y + 2z)}_{w}\,\mathbf{k}$

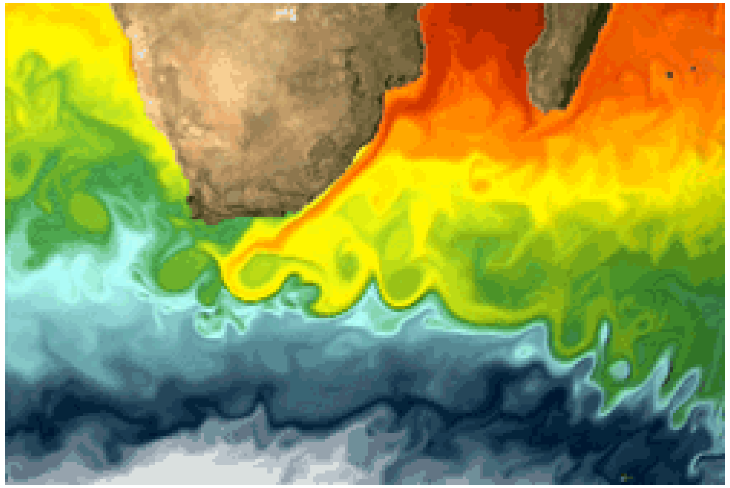
General continuity equation: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$

steady, incompressible (either)

- For incompressible steady: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$
- Evaluate the derivatives: $\frac{\partial u}{\partial x} = 4 \quad \frac{\partial v}{\partial y} = -3 \quad \frac{\partial w}{\partial z} = 2$
- Applying the continuity equations: $4 - 3 + 2 = 3 \neq 0$

Ans. (b): Thus, this flow is **not** possible. It does not conserve mass.





Computation Fluid Dynamics simulations of flow in the earth's oceans.
Source: http://the-science-llama.tumblr.com/post/31734297332/infinity-imagined-fluid-dynamics-of-earths

# END NOTES

Presentation prepared and delivered by Dr. David Naylor.