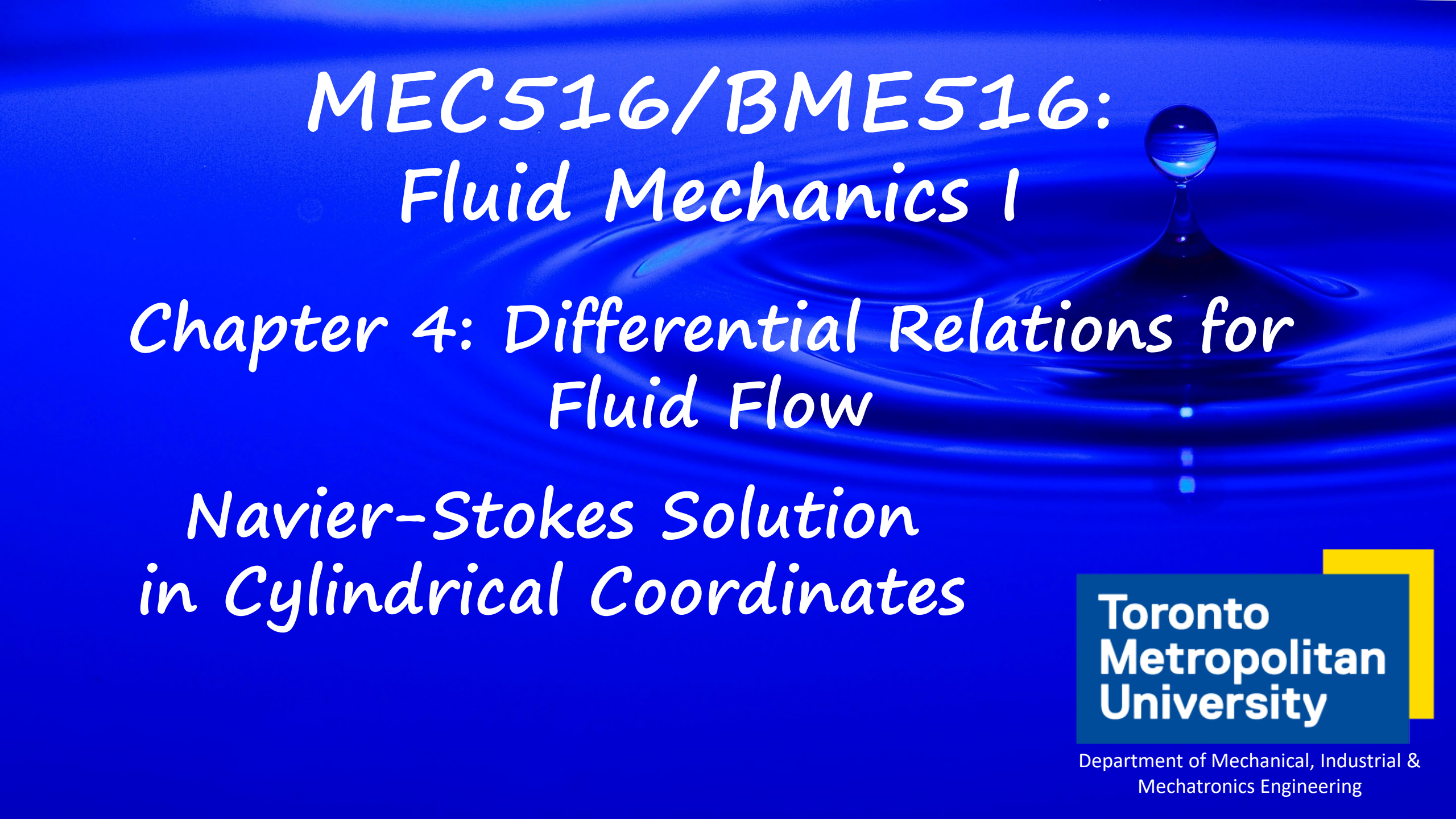
# MEC516/BME516: Fluid Mechanics I

## Chapter 4: Differential Relations for Fluid Flow

## Navier-Stokes Solution in Cylindrical Coordinates



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# Example: Navier-Stokes Solution for Flow in an Annulus

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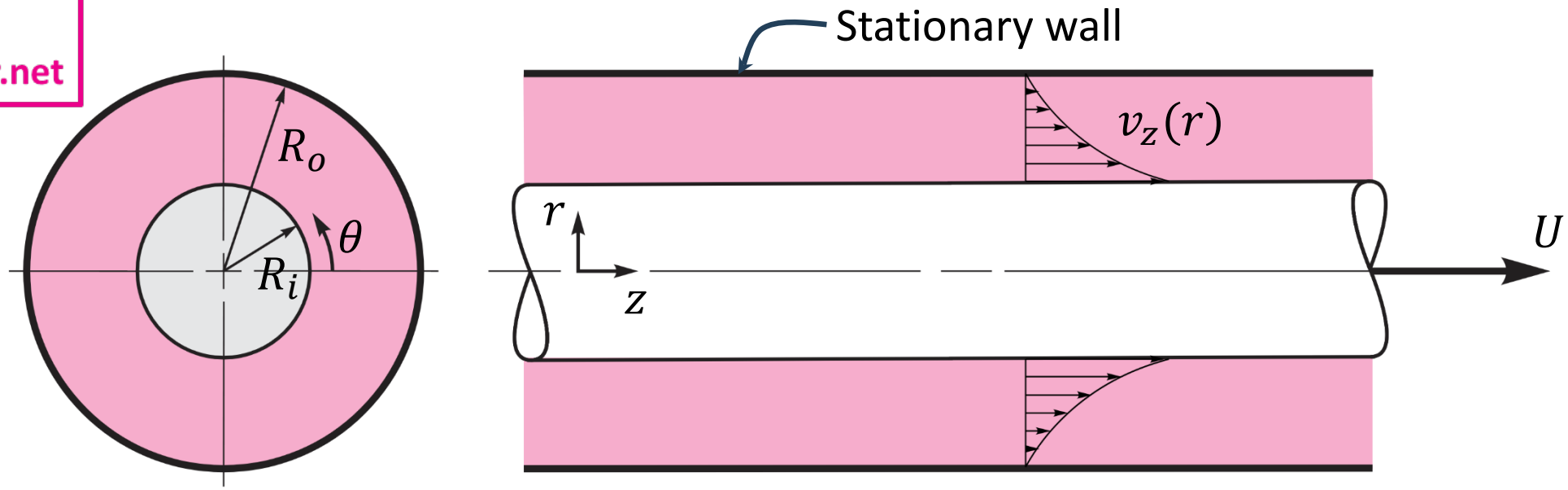
Consider steady laminar flow in a concentric annulus. A viscous incompressible fluid flows between a moving inner solid rod with radius $R_i$ and a stationary outer tube with radius $R_o$. The flow is driven entirely by the axial motion of an inner solid rod at velocity $U$. So, the axial pressure gradient $\partial p/\partial z = 0$. Far from inlet, the $r$-component of the fluid velocity is zero, $v_r = 0$. The flow is *axisymmetric* with no swirl, $v_\theta = 0$. Neglect the effects of gravity.

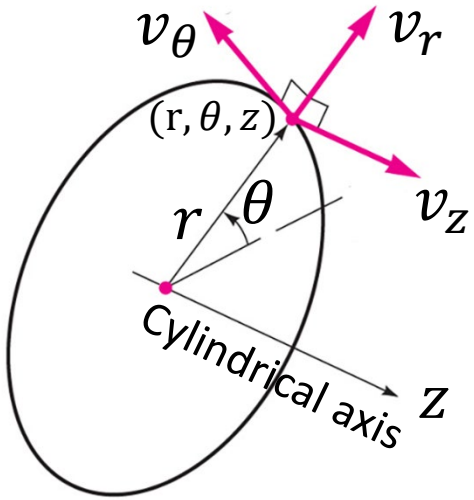
Derive an expression for axial velocity distribution of the fluid, $v_z(r)$.

PDFs available at
**www.drdavidnaylor.net**



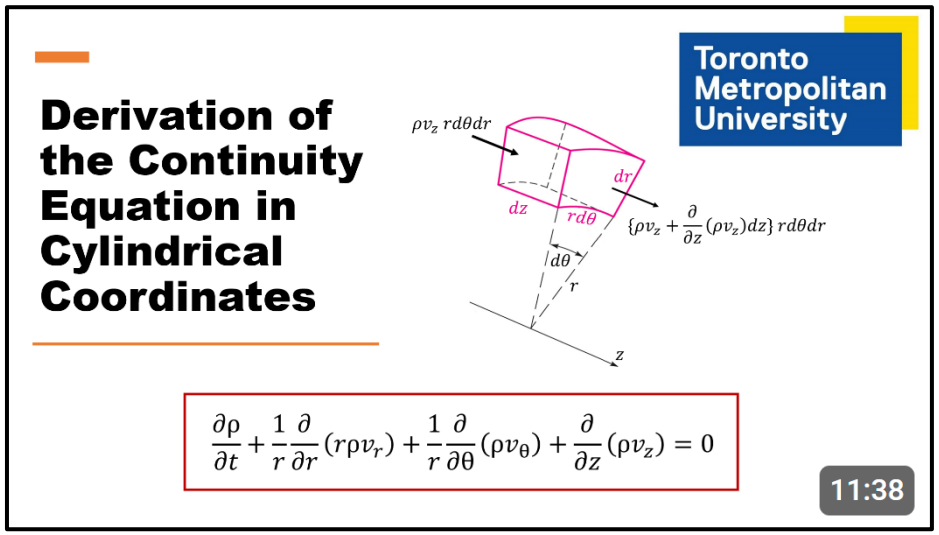
# Governing Equations

**Appendix D**
**Equations of Motion in Cylindrical Coordinates**




- Continuity equation for **incompressible** flow ($\rho = const.$)

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$




Derivation of the Continuity Equation in Cylindrical Coordinates

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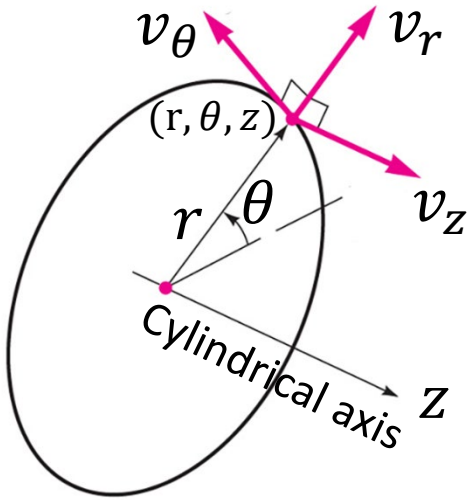
Fluid Matters

(Link in Video Description)

# Governing Equations

## Appendix D
## Equations of Motion in Cylindrical Coordinates




Conservation of momentum (incompressible Newtonian fluid, $\mu = const.$)

$$\frac{m\boldsymbol{a}}{\forall} = \frac{\boldsymbol{F}}{\forall}$$

$r$-direction:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

$\theta$-direction:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$

$z$-direction:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

← Only need z-equation

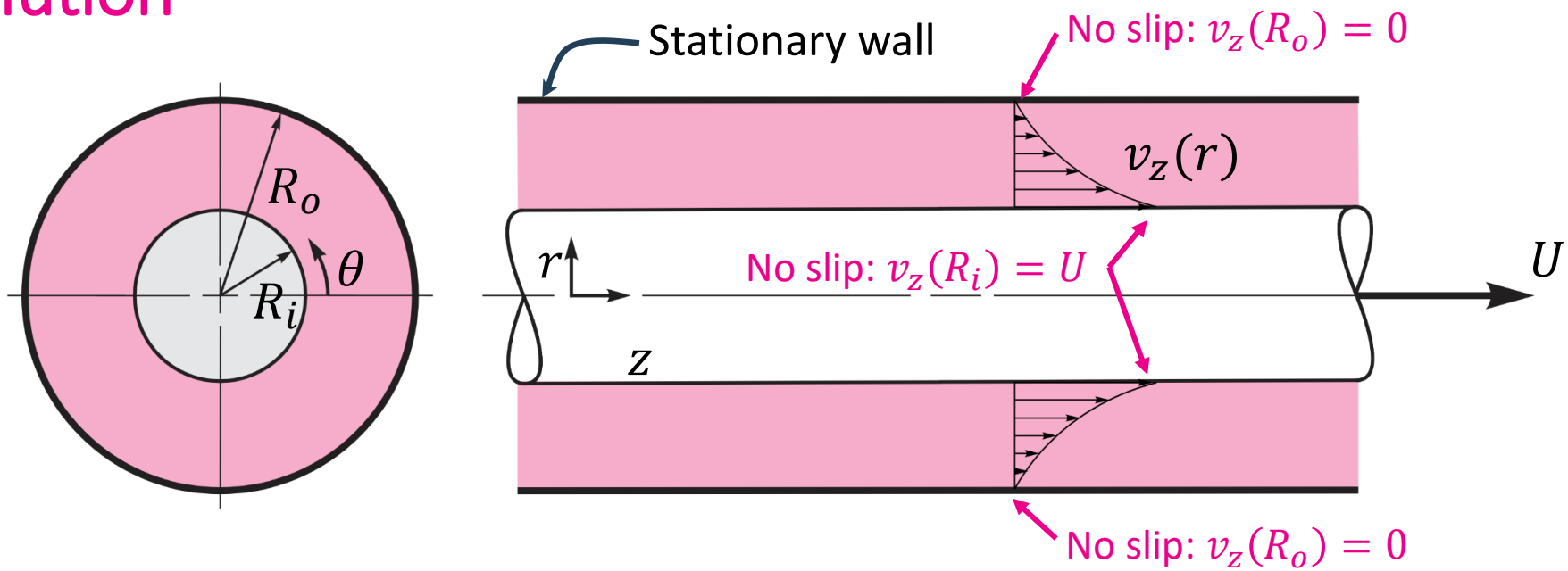
Local Acceleration

Convective Accelerations

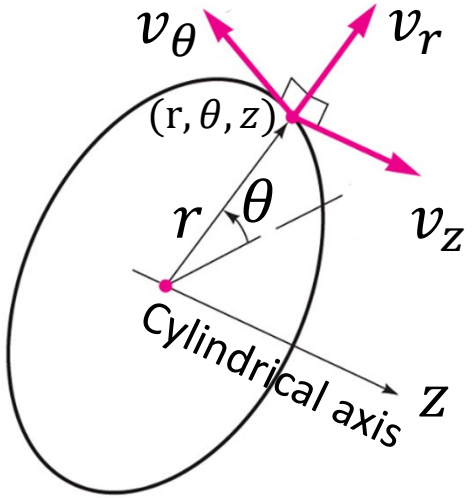
Pressure Gradient

Viscous Shear Forces

# Solution

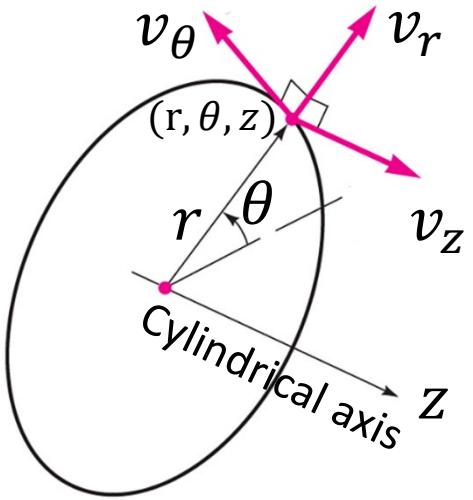



- From the problem statement:
  - No flow in the $r$- or $\theta$-directions: $v_r = v_\theta = 0$
  - Flow is wall-driven. No pressure gradient: $\frac{\partial p}{\partial z} = 0$
  - Flow is *axisymmetric*: $\frac{\partial *}{\partial \theta} = 0$
  - Boundary conditions at $r = R_i, R_o$
  - Neglect effects of gravity: $g_z = g_\theta = g_r = 0$



# Solution

- Continuity equation (conservation of mass):

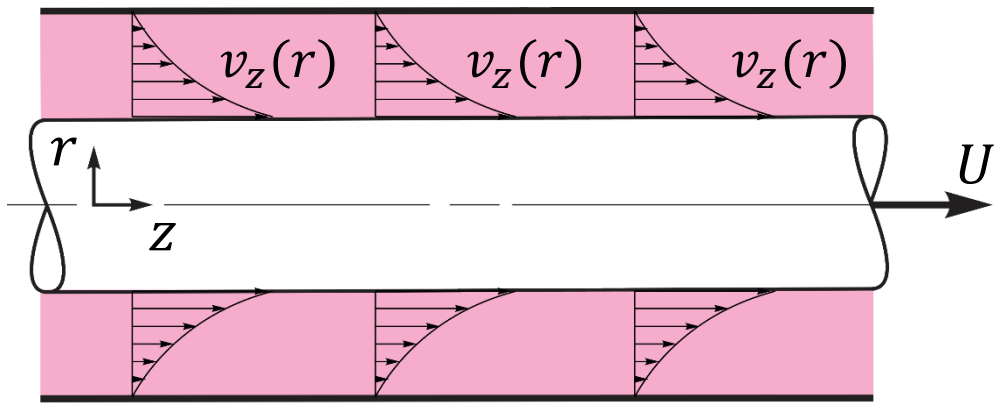



$v_r = 0$ $\quad$ $v_\theta = 0$ (No swirl & Axisym.)

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

$$\frac{\partial}{\partial z}(v_z) = 0$$

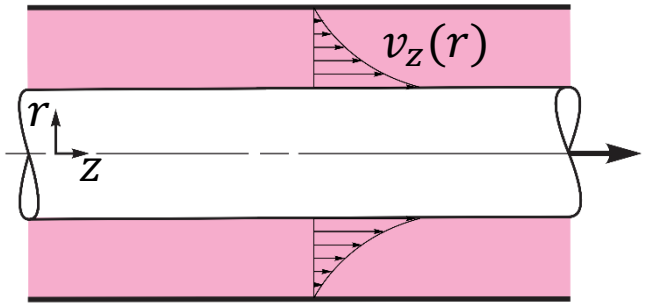
$v_z$ doesn't change axially

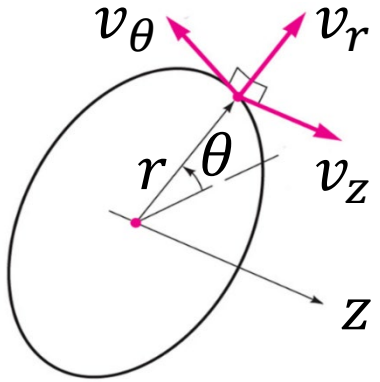



Thus, flow is *fully developed*

# Solution

Conservation of momentum (in z-direction):




Steady 0; $v_r = 0$; $v_\theta = 0$ & Axisym.; Fully Dev. 0; Wall-driven 0; Axisymmetric 0; Fully Dev. 0

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

Not Zero!

Note the full derivatives! PDE becomes an ODE because $v_z = v_z(r)\ only$

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

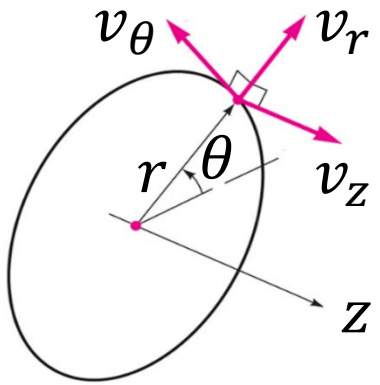
Multiply by $\frac{r}{\mu}$:

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

$$\therefore\ r\frac{dv_z}{dr} = C_1\ \text{(constant)}$$

# Solution

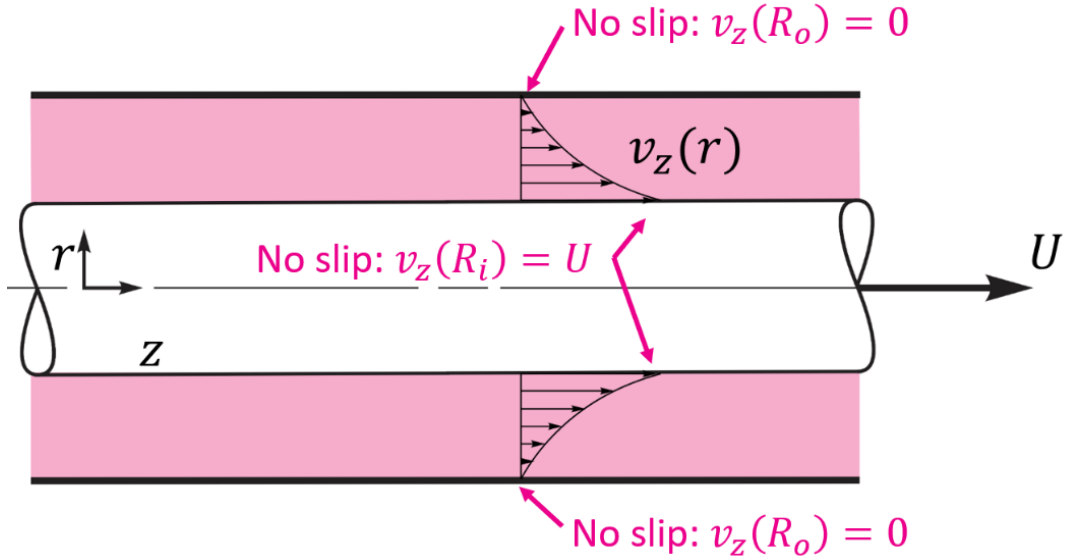
$$r\frac{dv_z}{dr} = C_1 \quad \text{(constant)}$$




Divide by r: $\frac{dv_z}{dr} = \frac{C_1}{r}$

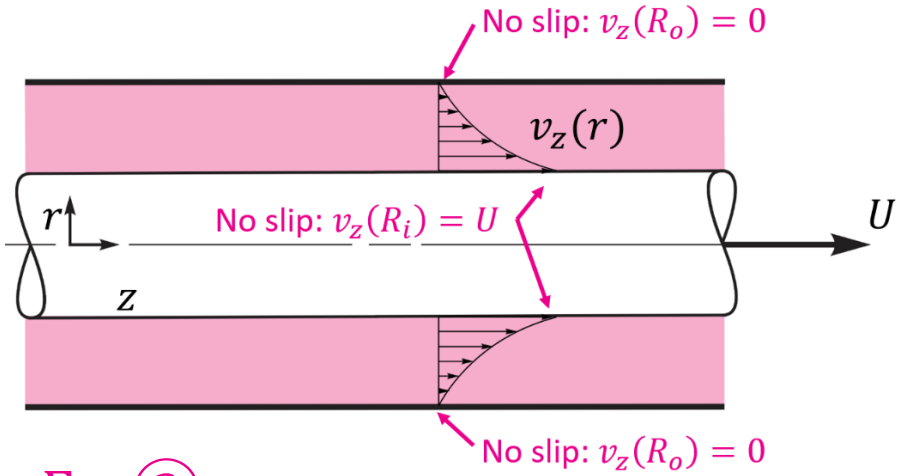
- Integrating: $v_z = C_1 \ln(r) + C_2$ Eq. ①

- Two boundary conditions (B.C.s) to determine $C_1$ and $C_2$



# Solution

$$v_z = C_1 \ln(r) + C_2 \qquad \text{Eq. } ①$$




- B.C. at fixed outer cylinder: $r = R_o, v_z = 0$

Using ①: $0 = C_1 \ln(R_o) + C_2 \quad \rightarrow \quad C_2 = -C_1 \ln(R_o)$ Eq. ②

- Sub. ② into ①: $v_z = C_1 \ln(r) - C_1 \ln(R_o) = -C_1 \ln\left(\frac{R_o}{r}\right)$ Eq. ③

*Recall:*

$\ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$

- B.C. at the inner rod: $r = R_i, v_z = U$

Using ③: $U = -C_1 \ln\left(\frac{R_o}{R_i}\right) \quad \rightarrow \quad C_1 = -\frac{U}{\ln\left(\frac{R_o}{R_i}\right)}$ Eq. ④

- Sub. ④ into ③:

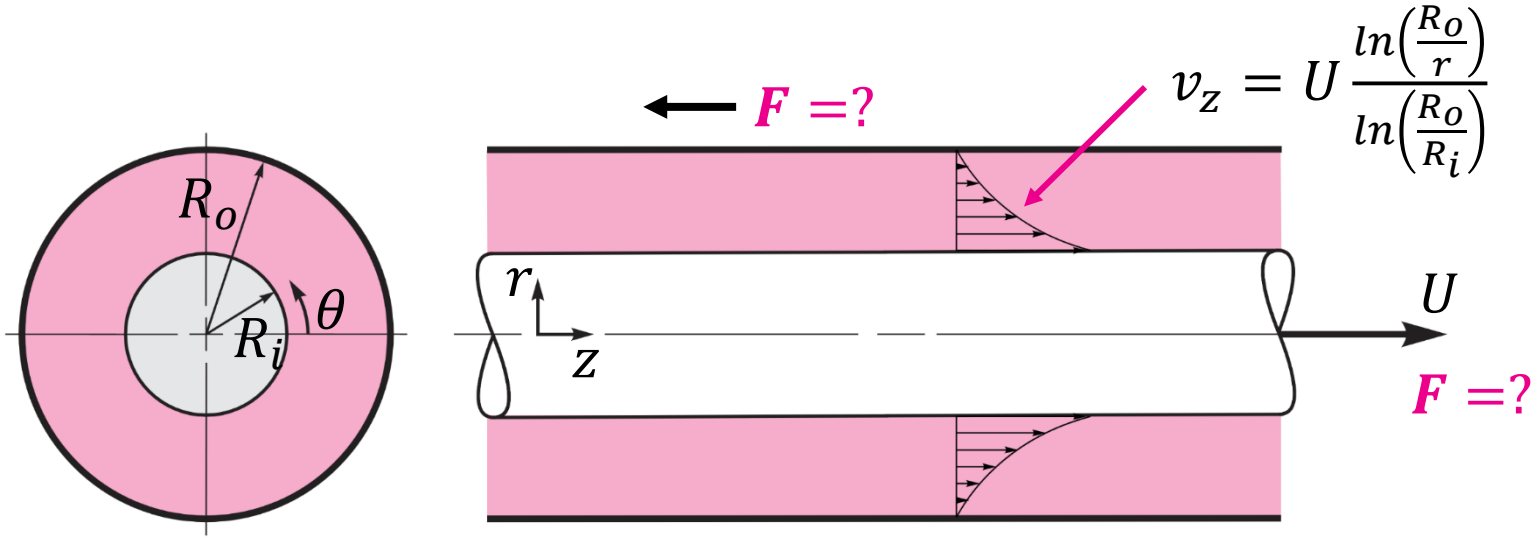
$$v_z = U \frac{ln\left(\frac{R_o}{r}\right)}{ln\left(\frac{R_o}{R_i}\right)}$$

***Ans.***

Check:

$r = R_o \quad v_z = 0$

$r = R_i \quad v_z = U$




Show that the force per unit length is:

$$\frac{F}{L} = \frac{2\pi\mu U}{\ln\left(\frac{R_o}{R_i}\right)}$$

Hint: $\frac{F}{L} = \frac{\tau A}{L}$ $where$ $\tau = \mu \frac{dv_z}{dr}\Big|_{r=R_i,R_o}$

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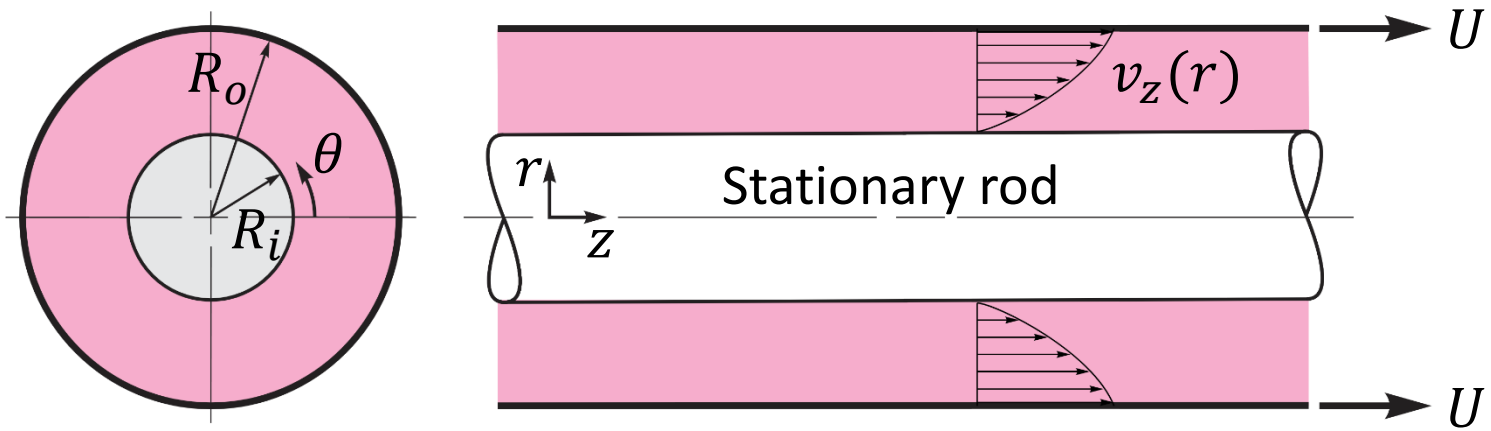
# A Variation: Moving Outer Cylinder and Stationary Inner Rod




Solution:

$$v_z = U \frac{ln\left(\frac{r}{R_i}\right)}{ln\left(\frac{R_o}{R_i}\right)}$$

Reference: White, F.M., “Viscous Fluid Flow”, McGraw-Hill, 1991.

## END NOTES

Presentation prepared and delivered by Professor David Naylor

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