MEC516/BME516: Fluid Mechanics I

Chapter 3: Control Volume Analysis Part 7

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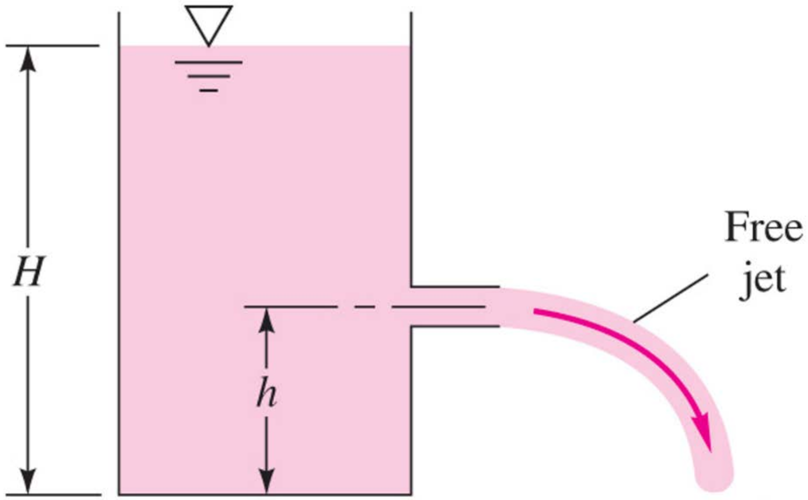
Overview

- Derivation of the Bernoulli Equation
  - powerful, but limited to steady, incompressible, frictionless flow

- Numerical Example
  - flow discharge from an open tank



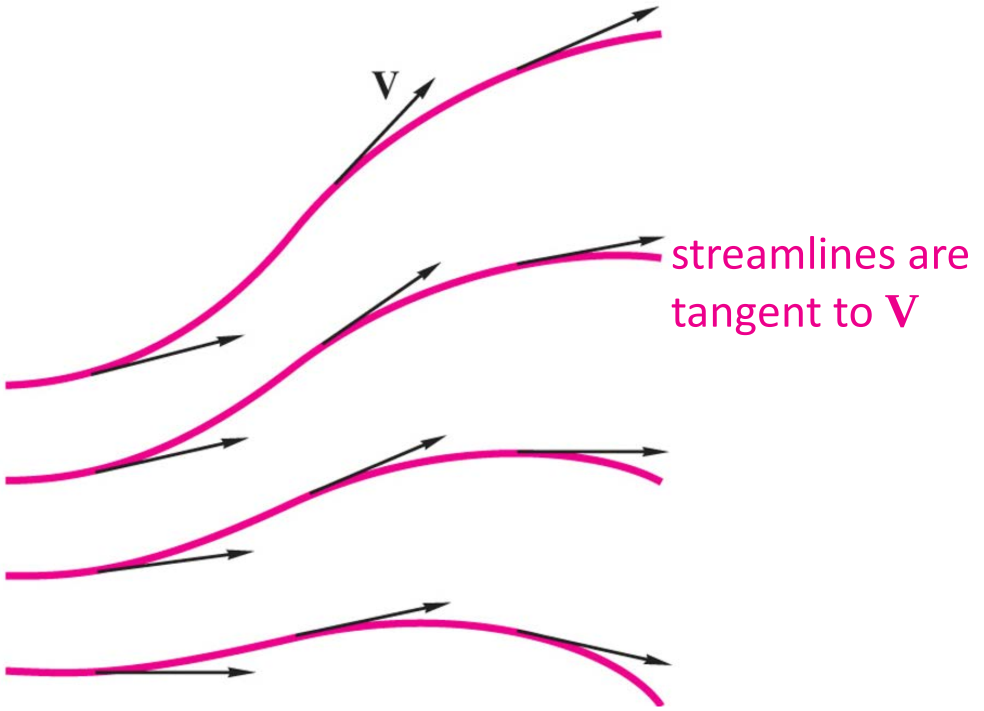
Daniel Bernoulli (1700-1782)

https://en.wikipedia.org/wiki/Daniel_Bernoulli

Derivation of the Bernoulli Equation (B.E.)

Assumptions (i.e. restrictions)

- steady flow
- incompressible flow (ρ=constant)
- flow along a *streamline*
  (Recall the video for Chapter 3 Part 1).
- frictionless flow, i.e. zero fluid viscosity, μ
  Ideal "*inviscid*" flow".

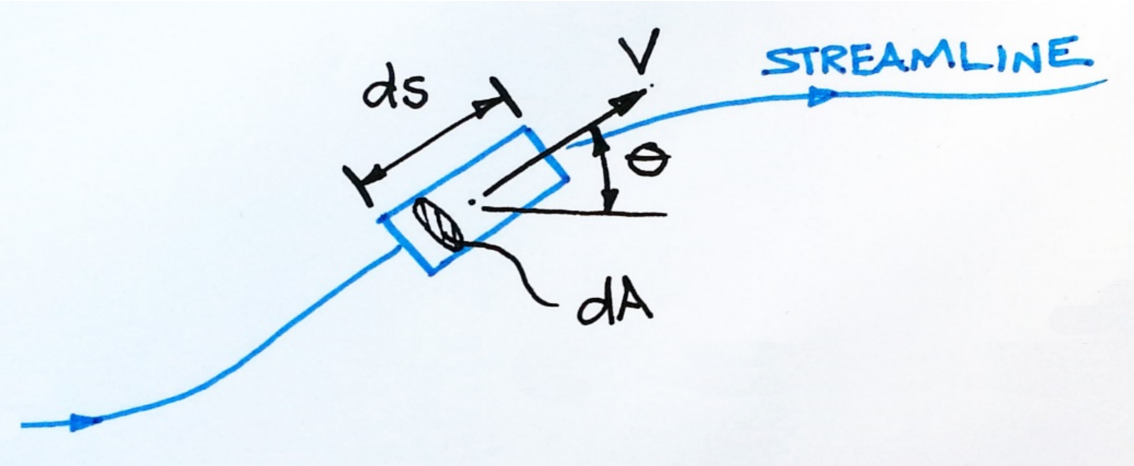


- All real fluids have viscosity. So, the B.E. is restricted to flow regions that are nearly frictionless.

- A detailed discussion of the limitations of B.E. will follow the derivation.

Derivation of the Bernoulli Equation

- Consider steady flow of a fluid "particle" along a streamline
  - "s" is the local streamline direction
  - the fluid particle has volume $d\forall$ and mass $\mathrm{dm} = \rho\, d\forall$



- Applying Newton's 2nd law for this *system*: (the fluid "particle", dm)

$$\Sigma \boldsymbol{F}_s = dm\, \boldsymbol{a}_s = \rho\, d\forall \frac{\partial V}{\partial t}$$

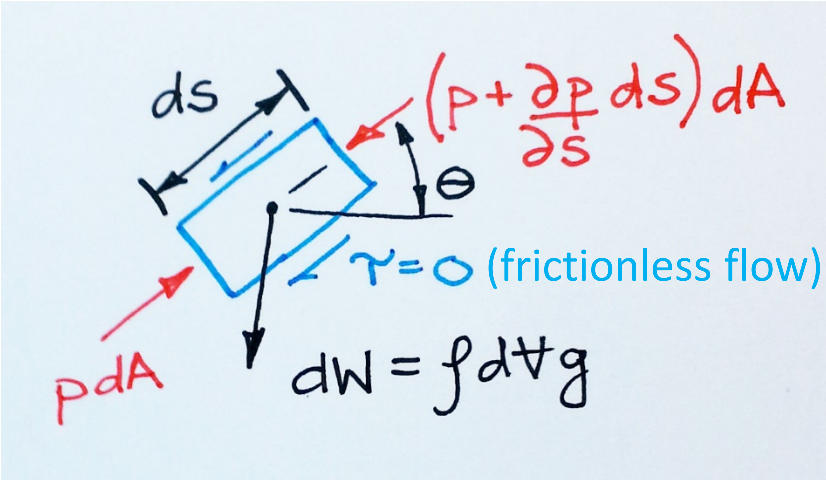
(How can the term $\frac{\partial V}{\partial t}$ exist if the flow is steady?)

Applying the chain rule: $\Sigma \boldsymbol{F}_s = \rho\, d\forall \underbrace{\frac{ds}{dt}\frac{\partial V}{\partial s}}_{\text{chain rule}}$ and noting that $\mathrm{V} = \frac{ds}{dt}$

We get: $$\Sigma \boldsymbol{F}_s = \rho\, d\forall\, V \frac{\partial V}{\partial s} = \rho\, d\forall \overbrace{\frac{1}{2}\frac{\partial (V^2)}{\partial s}}^{\text{rate of change of k.e. along streamline}}$$ Eq. (1)

Derivation of the Bernoulli Equation

- Now we consider the forces on the fluid particle.
  - fluid weight, $dW = dm\, g = \rho d\forall\, g$
  - pressure forces
  - no viscous shear stress, τ ! (Why not?)

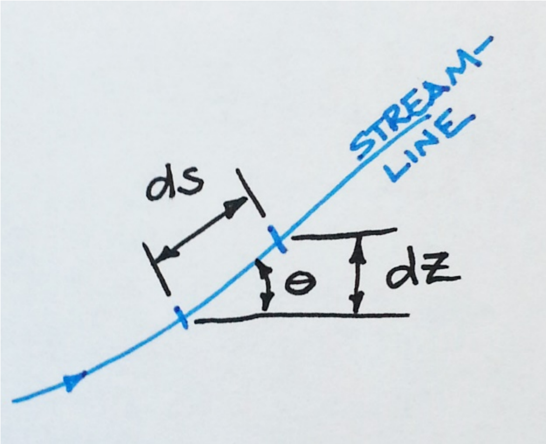


- Sum of the forces in the "s" direction:

$$\Sigma \boldsymbol{F}_s = pdA - \left(p + \frac{\partial p}{\partial s}\, ds\right) dA - \rho d\forall\, g \sin\theta$$

Simplifying: $$\Sigma \boldsymbol{F}_s = -\frac{\partial p}{\partial s}\, ds\, dA - \rho d\forall\, g \sin\theta$$

Note that: $ds\, dA = d\forall$ and $\sin\theta = \dfrac{dz}{ds}$ (z is altitude)



So, we get: $$\Sigma \boldsymbol{F}_s = -\frac{\partial p}{\partial s}\, d\forall - \rho d\forall\, g \frac{dz}{ds}$$ Eq. (2)

Derivation of the Bernoulli Equation

Newton's 2nd law: $$\Sigma \boldsymbol{F}_s = \rho\, d\forall\, \frac{1}{2}\, \frac{\partial (V^2)}{\partial s}$$ Eq. (1)

Forces on fluid particle: $$\Sigma \boldsymbol{F}_s = -\frac{\partial p}{\partial s}\, d\forall\; - \rho d\forall\, g\, \frac{dz}{ds}$$ Eq. (2)

Equate Eq. (1) and Eq. (2):

$$\rho\, d\forall\, \frac{1}{2}\, \frac{\partial (V^2)}{\partial s} = -\frac{\partial p}{\partial s}\, d\forall\; - \rho d\forall\, g\, \frac{dz}{ds}$$

Simplifying and dropping the partial derivatives (s is the only dependent variable):

$$\frac{1}{2}\, \frac{d(V^2)}{ds} + \frac{1}{\rho}\frac{dp}{ds} + g\, \frac{dz}{ds} = 0$$

Thus, along the streamline: $$\frac{1}{2}\; d(V^2) + \frac{1}{\rho}\, dp\; + g\, dz = 0$$

Derivation of the Bernoulli Equation

Re-written: $\frac{1}{2}\ d(V^2) + \frac{1}{\rho}\ dp\ + g\ dz = 0$

We can now integrate this equation along the streamline:

$$\int \frac{1}{2}\ d(V^2) + \int \frac{1}{\rho}\ dp + \int g\ dz = 0$$

Integrating:

$$\boxed{\frac{V^2}{2} + \frac{p}{\rho} + gz = const}$$

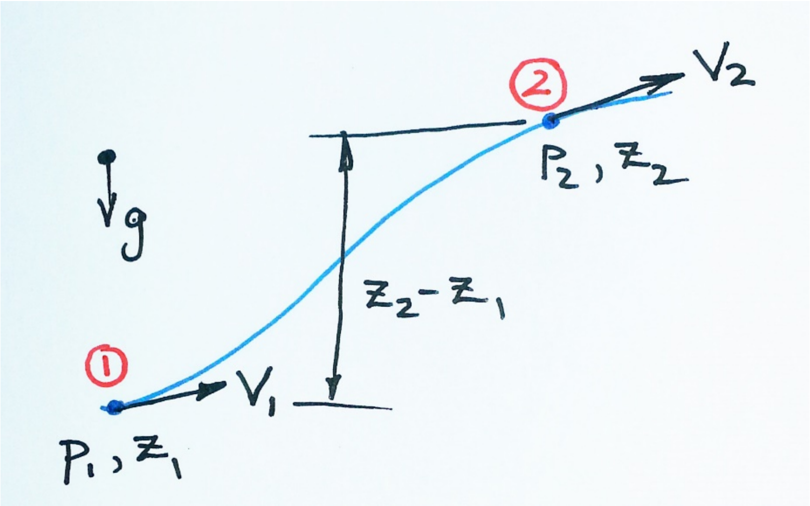
(on a streamline)

This is the famous ***Bernoulli Equation***.

Derivation of the Bernoulli Equation

Alternately, integration can be made between two points (1) and (2) on the streamline:

$$\int_{V_1}^{V_2} \frac{1}{2}\, d(V^2) + \int_{p_1}^{p_2} \frac{1}{\rho}\, dp + \int_{z_1}^{z_2} g\, dz = 0$$



Integrating:

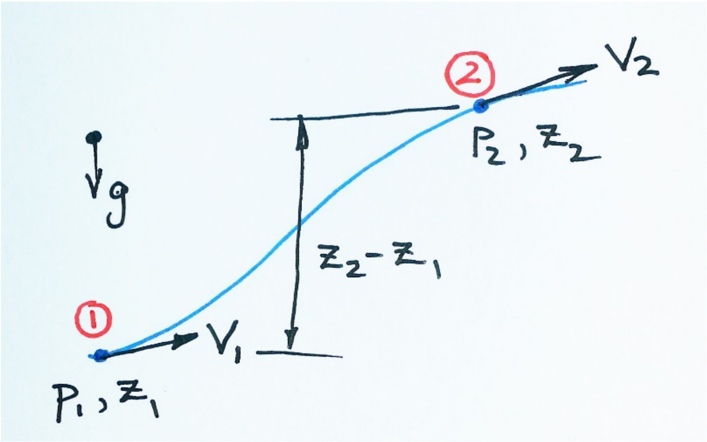
$$\frac{1}{2}(V_2^2 - V_1^2) + \left(\frac{p_2}{\rho} - \frac{p_1}{\rho}\right) + (gz_2 - gz_1) = 0$$

$$\boxed{\frac{1}{2}V_1^2 + \frac{p_1}{\rho} + gz_1 = \frac{1}{2}V_2^2 + \frac{p_2}{\rho} + gz_2}$$ (on a streamline)

This is an alternate form of the Bernoulli equation. We will use this form most frequently.

Interpretation of the Bernoulli Equation

- B.E. can be interpreted as an energy balance
- For frictionless flow, there is no energy loss caused by viscous dissipation.
- Total energy in the flow remains constant.



$$\overbrace{\underbrace{\frac{1}{2}V_1^2}_{\substack{\text{kinetic energy}\\ \text{per unit mass}\\ \text{at point (1)}}} + \underbrace{\frac{p_1}{\rho}}_{\substack{\text{pressure energy}\\ \text{per unit mass}\\ \text{at point (1)}}} + \underbrace{g z_1}_{\substack{\text{potential energy}\\ \text{per unit mass}\\ \text{at point (1)}}}}^{\substack{\text{Total energy per unit mass}\\ \text{at point (1)}}} = \overbrace{\underbrace{\frac{1}{2}V_2^2}_{\substack{\text{kinetic energy}\\ \text{per unit mass}\\ \text{at point (2)}}} + \underbrace{\frac{p_2}{\rho}}_{\substack{\text{pressure energy}\\ \text{per unit mass}\\ \text{at point (2)}}} + \underbrace{g z_2}_{\substack{\text{potential energy}\\ \text{per unit mass}\\ \text{at point (2)}}}}^{\substack{\text{Total energy per unit mass}\\ \text{at point (2)}}}$$

The Fluid Mechanics Definition of "Head"

- If we divide the Bernoulli equation by "g" we get another form:

$$\frac{1}{2g}V^2 + \frac{p}{\gamma} + z = const$$

- Note that $\gamma = \rho\, g$, the specific weight of the fluid (N/m^3)
- All the terms have units of height (m, ft) or "head". (Dimensionally homogeneous)

$\frac{1}{2g}V_1^2$ is called the *velocity head,* (m) (ft)

$\frac{p_1}{\gamma}$ is called the *pressure head,* (m) (ft)

z_1 is called the *elevation head,* (m) (ft)

The Fluid Mechanics Definition of “Head”

- Pump specifications use this terminology.

Liberty Pumps®

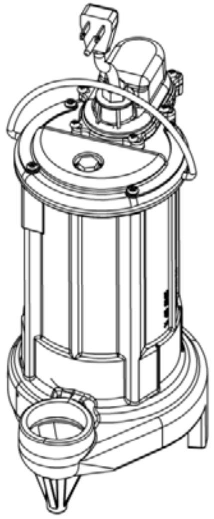
Pump Specifications

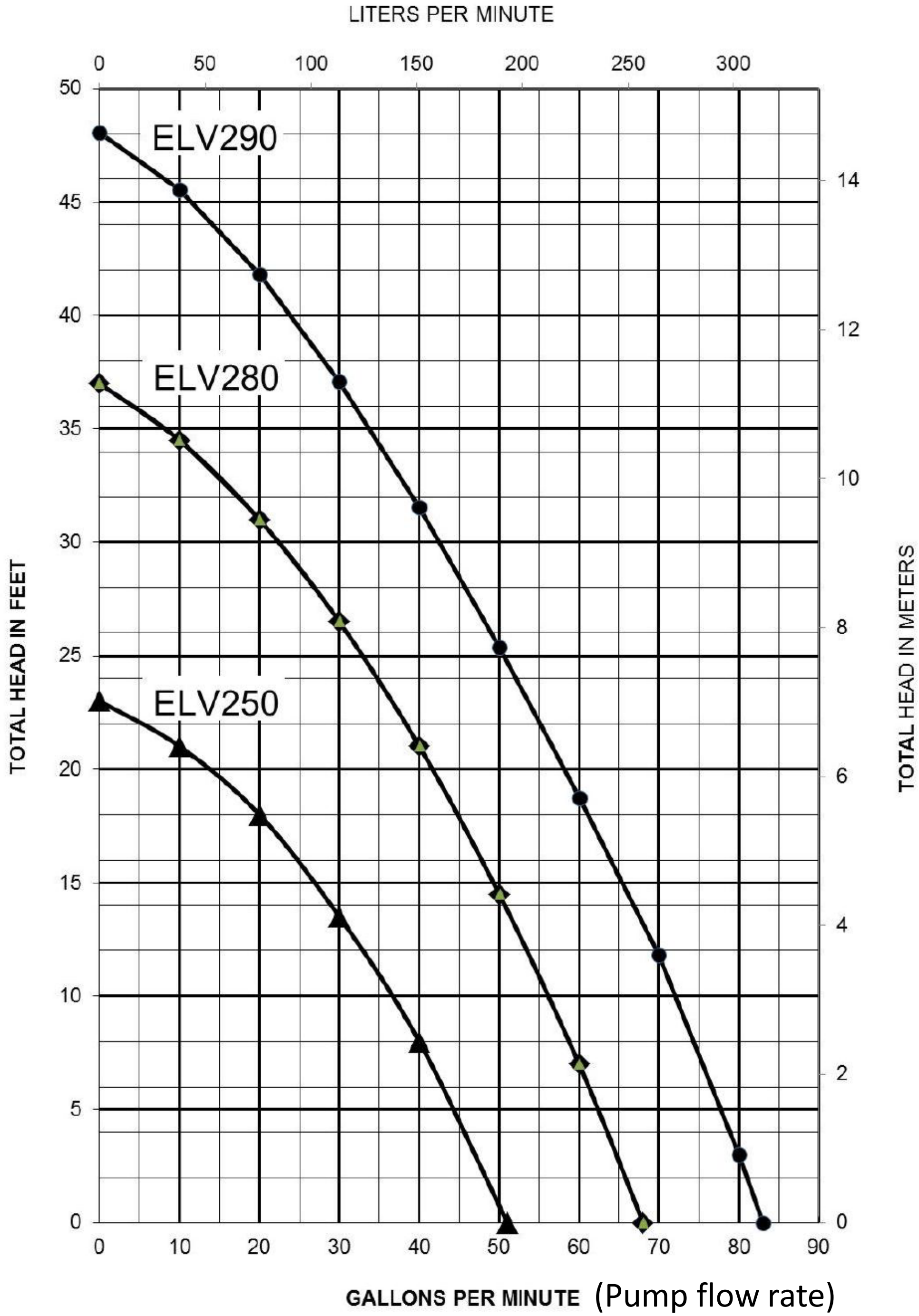
ELV Series Submersible Sump Pump with OilTector Control

ELV250 1/3 hp

ELV280 1/2 hp

ELV290 3/4 hp





source: http://www.libertypumps.com/

Limitations on the Use of the Bernoulli Equation

1. Steady flow.
2. Incompressible flow. Good for liquid flows. Gas flows where compressibility effects are small. Recall our "rule of thumb", Ma<0.3. For air near room temp.

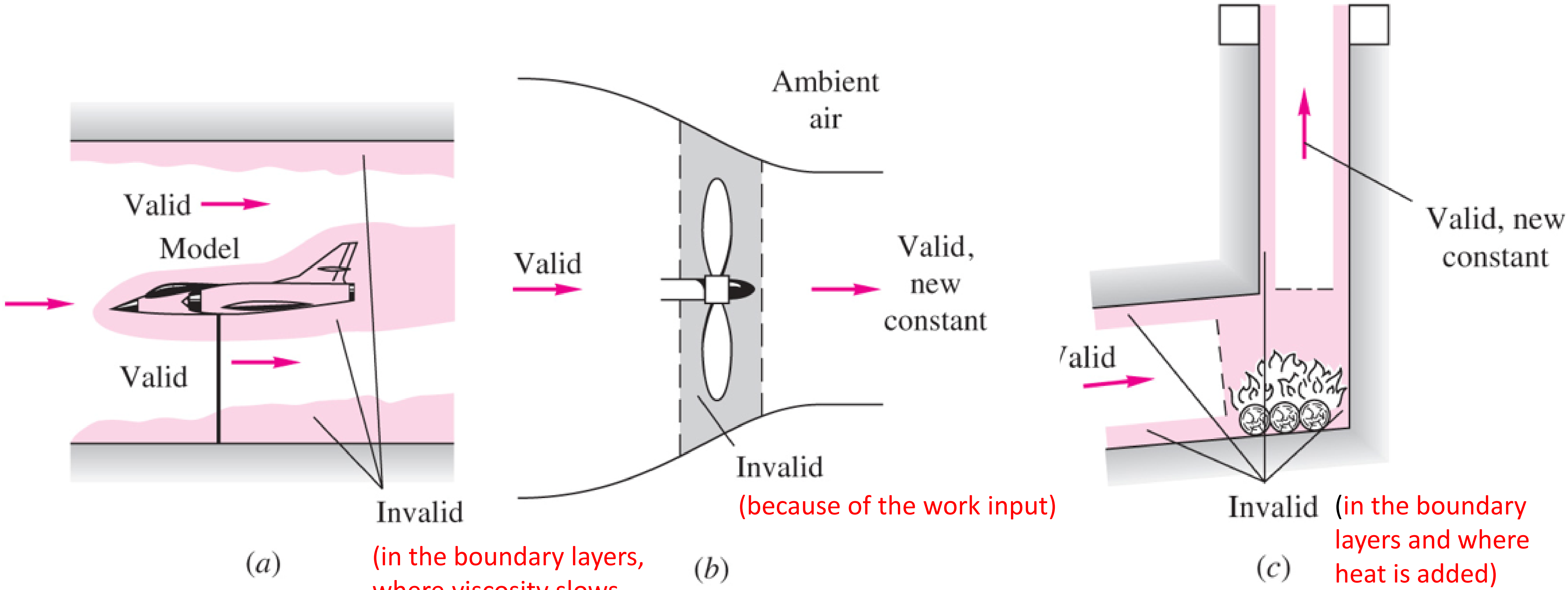
$$V < 0.3c = 0.3\sqrt{kRT} = 0.3\sqrt{1.4\left(287\frac{J}{kgK}\right)300K} \approx 100\frac{m}{s} \quad (370\frac{km}{hr})$$

3. Frictionless flow. Bernoulli equation does **<u>not</u>** apply:
   - near solid walls where viscous effects dominate.
   - in regions of intense fluid mixing where energy is lost, e.g. flow throttled across a valve.

4. Flow along a streamline. In some situations, different streamlines may have different constants.

5. No mechanical device (pumps or turbines) between the two points on the streamline where B.E. is applied. No addition or extraction of heat/work allowed, since energy is constant along a streamline. Isothermal flows.

Limitations on the Use of the Bernoulli Equation



(in the boundary layers, where viscosity slows down the flow)

(because of the work input)

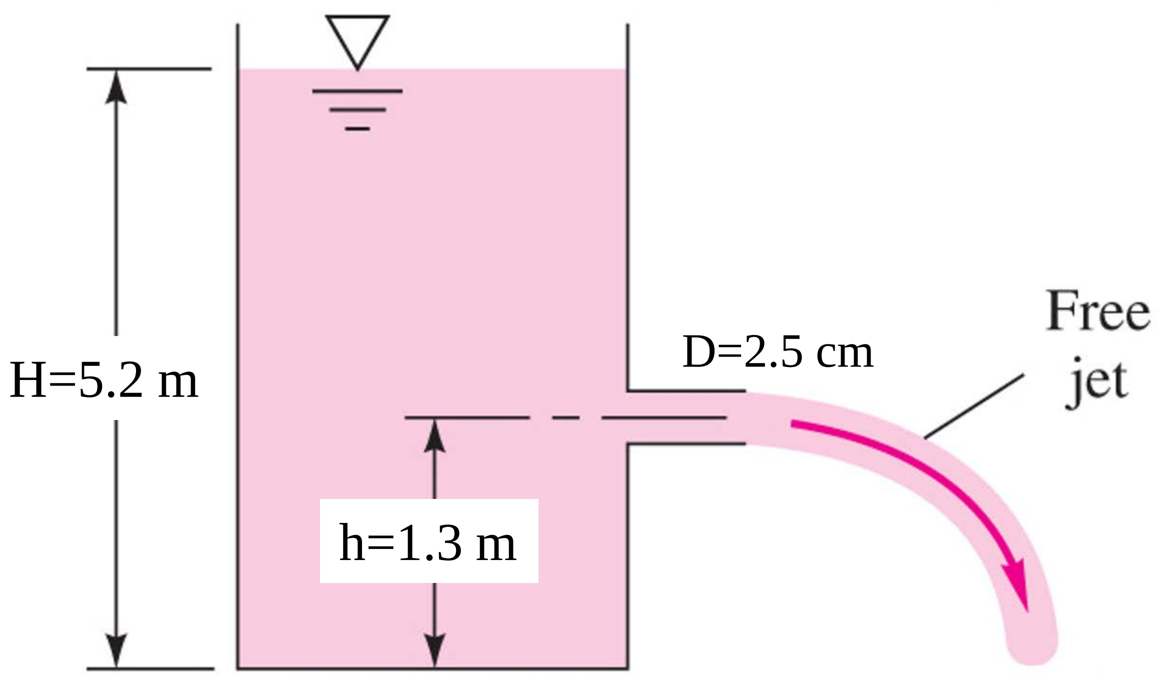
(in the boundary layers and where heat is added)

The viscous boundary layers are much thinner than shown here (typically of order ~1 cm).

Example

Water (at 20 °C) in a large diameter tank discharges through a small pipe. The height of the tank is 5.2 m. The discharge pipe is located 1.3 m from the bottom of the tank. The discharge pipe has an inside diameter of 2.5cm. The tank drains slowly, such that the flow can be approximated as steady and frictionless.

Calculate the flow rate from the tank (Q) in liters per second.

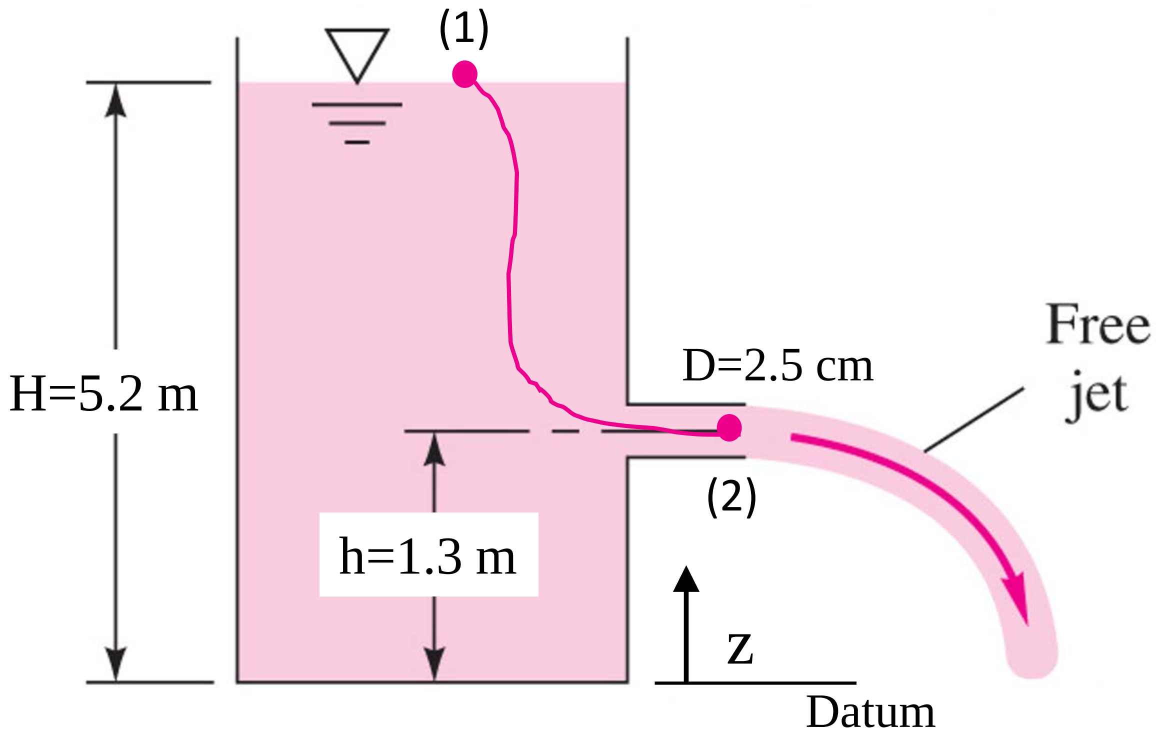


Example

Start by drawing a streamline.

- Point (1) is on the free surface of the tank.
- Point (2) is at the pipe discharge.

(This is arbitrary. Other points are possible.)



Apply the Bernoulli equation between (1) and (2):

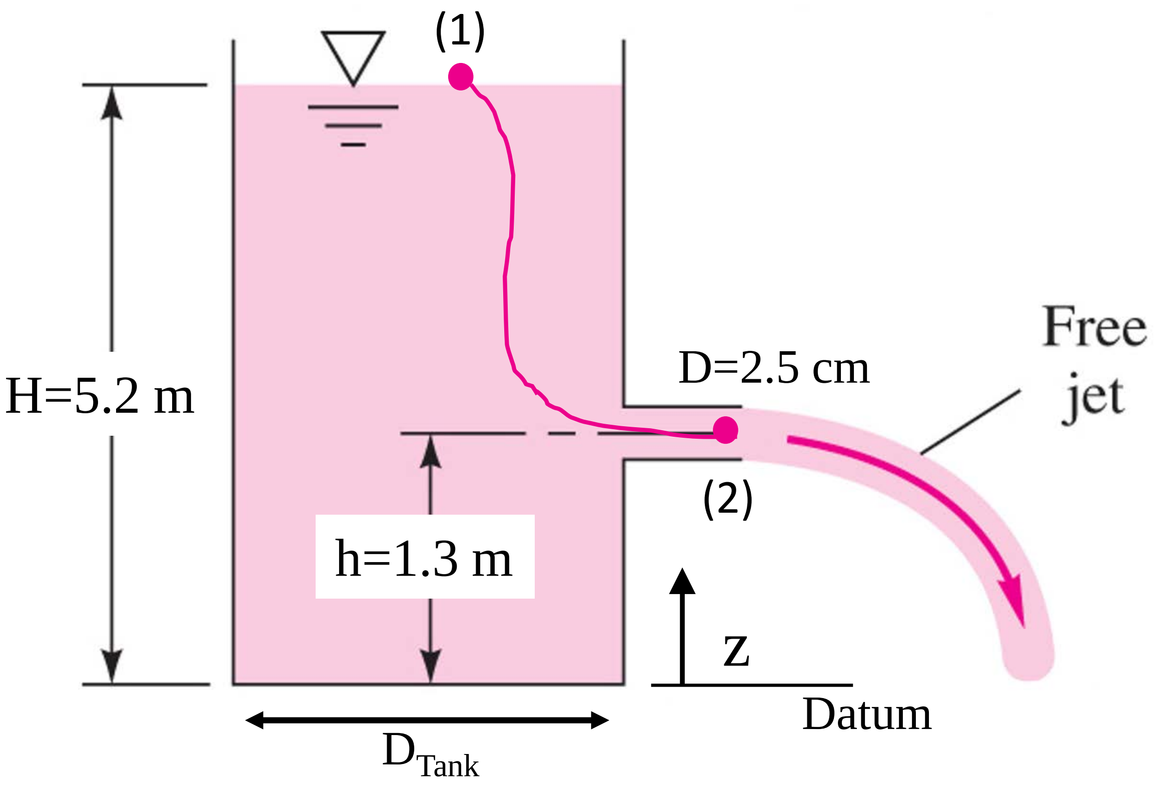
$$\frac{1}{2}V_1^2 + \cancel{\frac{p_1}{\rho}} + gz_1 = \frac{1}{2}V_2^2 + \cancel{\frac{p_2}{\rho}} + gz_2$$

Recall that the pressure at the pipe discharge is atmospheric pressure. So, $p_1 = p_2 = p_{atm}$

Example

$$\frac{1}{2}\cancel{V_1^2} + gz_1 = \frac{1}{2}V_2^2 + gz_2$$

0



The diameter of the tank, D_{Tank} , is large.

Thus, $V_1 \approx 0$

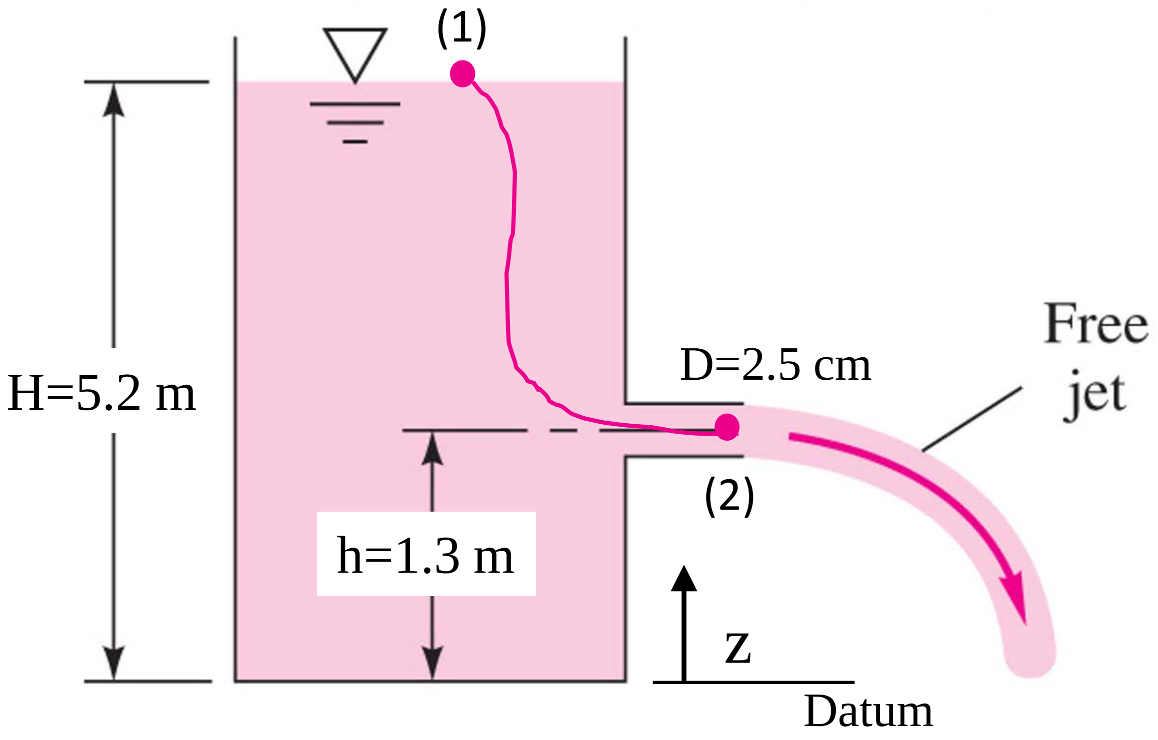
The water level falls slowly -- can neglect k.e. at (1)

Measuring z from the bottom of the tank: $z_1 = 5.2\,m \quad z_2 = 1.3\,m$

So, the Bernoulli equation gives: $V_2 = \sqrt{2g\,(z_1 - z_2)} = \sqrt{2\,\left(9.81\,\frac{m}{s^2}\right)(5.2 - 1.3)m} = \;\; 8.75\,\frac{m}{s}$

This is the same as water in freefall. No losses. Potential energy at (1) converted to kinetic energy at (2).

Example



We have: $V_2 = 8.75\ \frac{m}{s}$

Thus, the discharge flow rate is: $Q = V_2 A_2 = V_2\ \pi \frac{D_2^2}{4} = \frac{8.75m}{s}\ \pi \frac{(0.025m)^2}{4} = 0.00429\ \frac{m^3}{s} = 4.29\ l/s$

(Will the actual flow likely be higher or lower? Why?)

Ans.

Streaklines (streamlines) over a high performance car. The Bernoulli equation would apply for this flow, except very near the surface.
Source: http://6speedhaven.tumblr.com/post/25448453293

END NOTES

Presentation prepared and delivered by Dr. David Naylor.