MEC516/BME516: Fluid Mechanics I

Chapter 2: Fluid Statics

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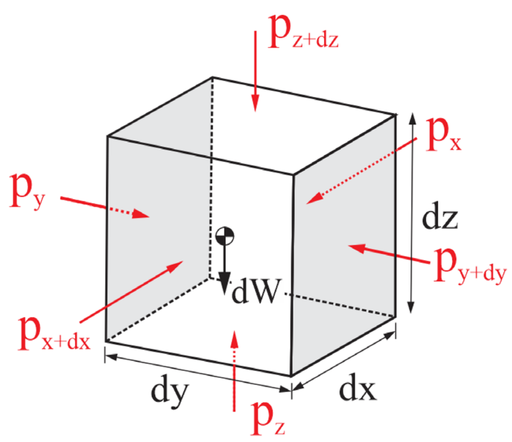
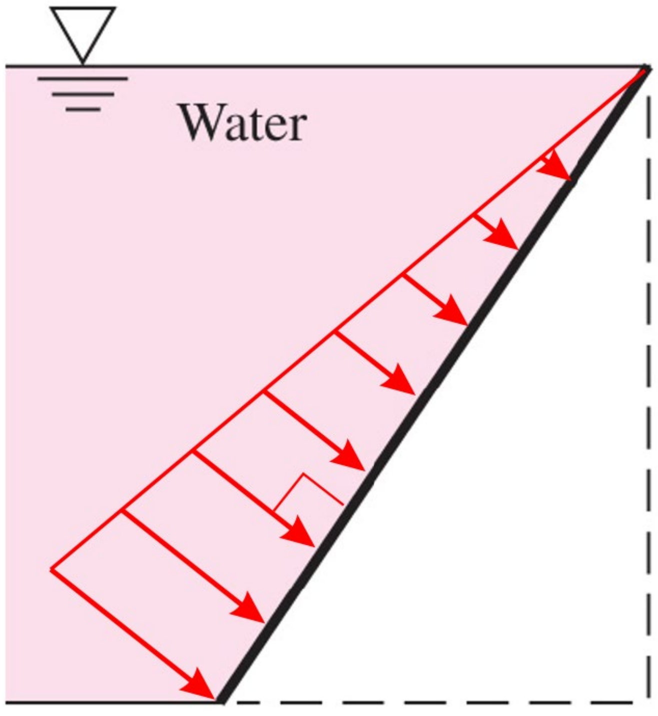
Overview

- Pressure distribution in a static fluid
  - Incompressible fluids (liquids)
  - Compressible fluids (gases)





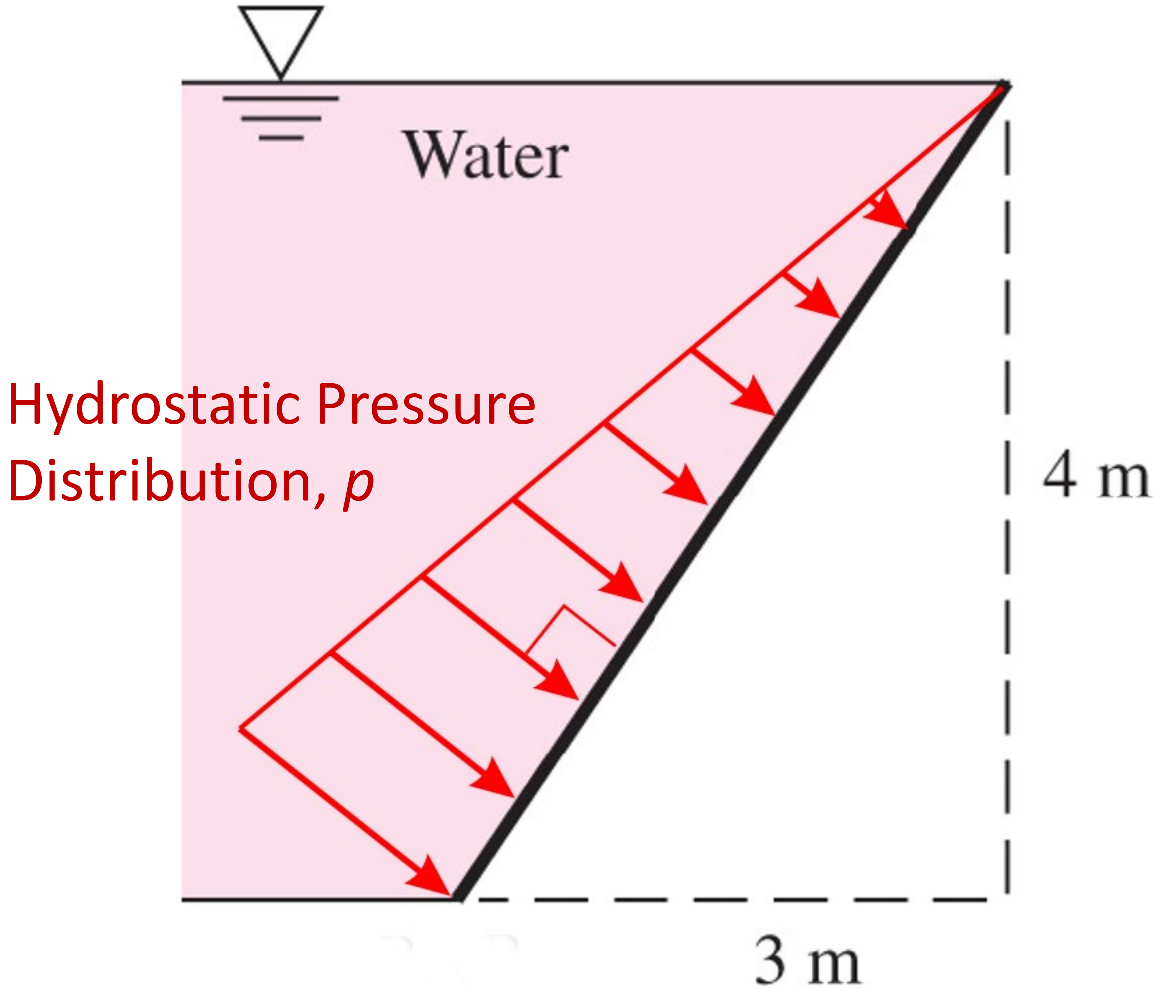
- Measurement of Pressure
  - Absolute and Gauge Pressure
  - Bourdon tube gauge
  - Mercury barometer

Pressure Distribution in a Static Fluid

- Pressure is the normal stress in a static fluid, i.e. force per unit area

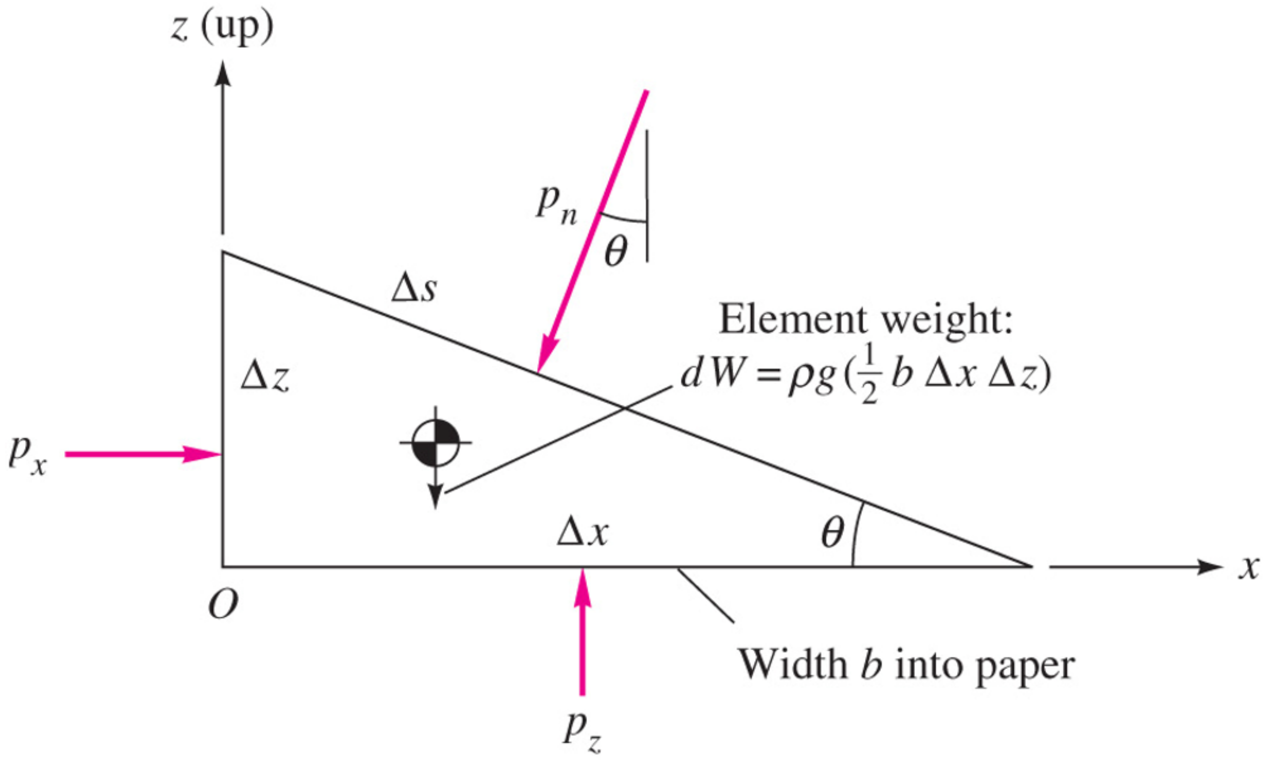
$$p = \frac{F_{normal}}{A} \qquad (\text{N/m}^2\text{, Pa})$$

- Acts **normal** to a surface



Pressure at a Point

- Consider pressure forces on wedge of static fluid with arbitrary angle, θ
- Using $\sum F_x = \sum F_z = 0$, can easily show that:



Result:

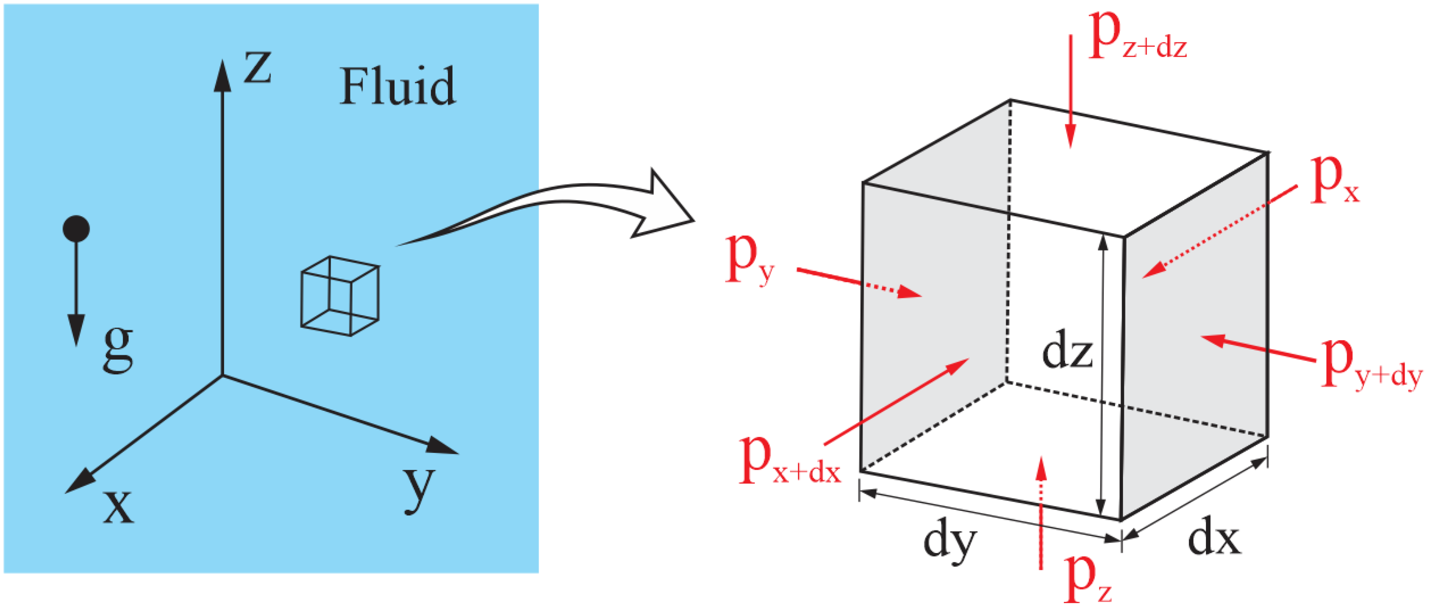
$$p_x = p_z = p_n = p \quad \text{(static fluid)}$$

Pressure acts equally in all directions at a point

Hydrostatic Pressure Distribution in a Static Fluid

- Consider a force balance. Since the fluid is stationary:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$



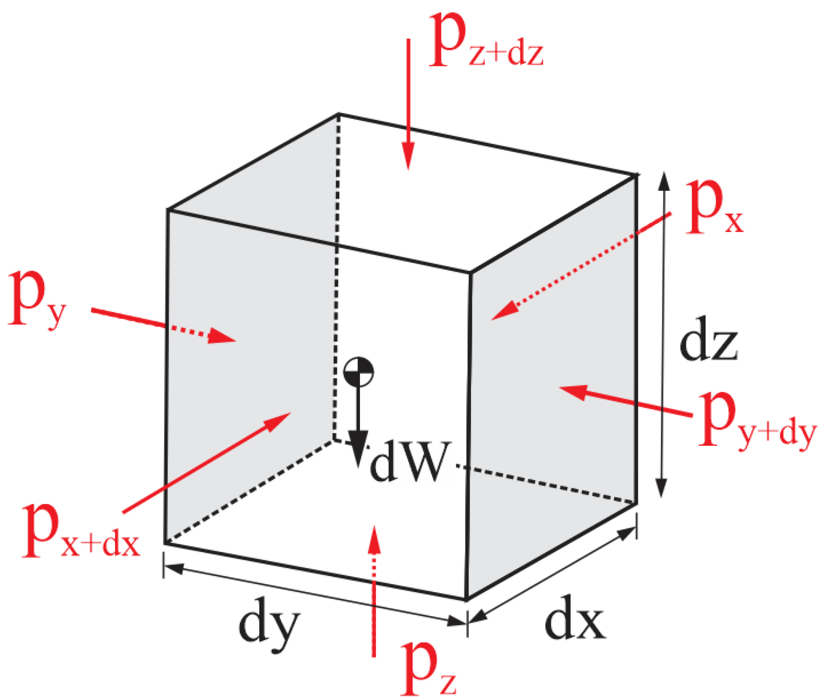
Hydrostatic Pressure Distribution in a Static Fluid

- Since the fluid is stationary:

$$\sum F_x = p_x dydz - p_{x+dx} dydz = 0$$

$$\sum F_y = p_y dxdz - p_{y+dy} dxdz = 0$$

$$\sum F_z = p_z dxdy - p_{z+dz} dxdy - \underbrace{dW}_{\text{weight}} = 0$$



Note: $dW = gdm = g\rho dV = g\rho dxdydz$

Taylor expansion:

$$p_{x+dx} = p_x + \frac{\partial p}{\partial x} dx$$

Note the partial derivative, because *p=p(x,y,z)*

Hydrostatic Pressure Distribution in a Static Fluid

- Making the substitution into the x-equation:

$$\sum F_x = p_x dydz - (p_x + \frac{\partial p}{\partial x} dx) dydz = 0$$

$$p_x dydz = (p_x + \frac{\partial p}{\partial x} dx) dydz \rightarrow \frac{\partial p}{\partial x} = 0 \quad \text{Thus,} \, p \neq p(x)$$

- Pressure is constant in the x direction. Similarly, for the y-direction:

$$\frac{\partial p}{\partial y} = 0 \quad \text{Thus,} \; p \neq p(y)$$

- Pressure is constant in the y direction

Hydrostatic Pressure Distribution in a Static Fluid

- Now in making the substitution in the z-direction:

$$\sum F_Z = p_z dxdy - \left(p_z + \frac{\partial p}{\partial z} dz\right) dxdy - g\rho dxdydz = 0$$

Gives $\frac{\partial p}{\partial z} = -\rho g$ → Use full derivative $\frac{dp}{dz} = -\rho g$ since $p \neq p(x, y), p = p(z)$ only

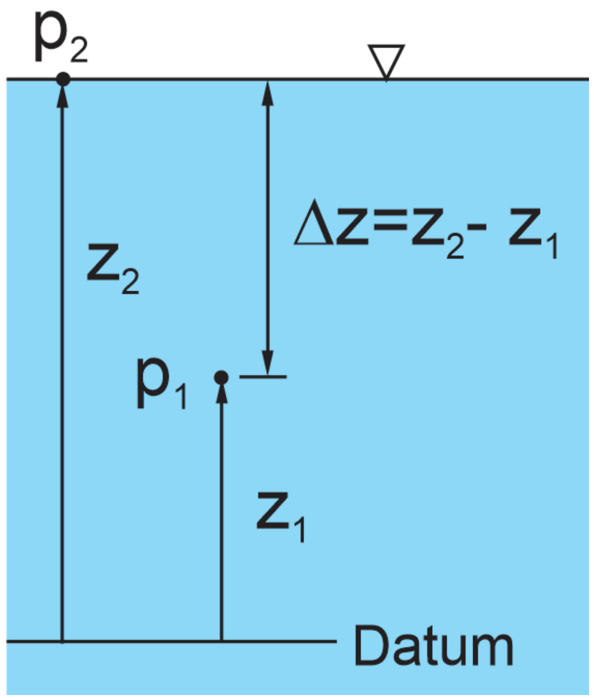
For ρ=constant, integrating gives: $p(z) = -\rho g z + C_1$

- Thus, pressure varies linearly in the z-direction for an incompressible fluid

Hydrostatic Pressure Distribution in a Static Fluid

- We have shown that: $\frac{dp}{dz} = -\rho g$ (variable density)

- We can integrate this for an incompressible fluid (ρ=const.):



$$\int_{p_1}^{p_2} dp = -\rho g \int_{z_1}^{z_2} dz$$

$$p_2 - p_1 = -\rho g(z_2 - z_1)$$

$$p_1 = p_2 + \rho g \Delta z$$

For an incompressible fluid: $p_1 = p_2 + \gamma \Delta z$

$\gamma = \rho g$ is specific weight

Hydrostatic Pressure Distribution in a Static Fluid

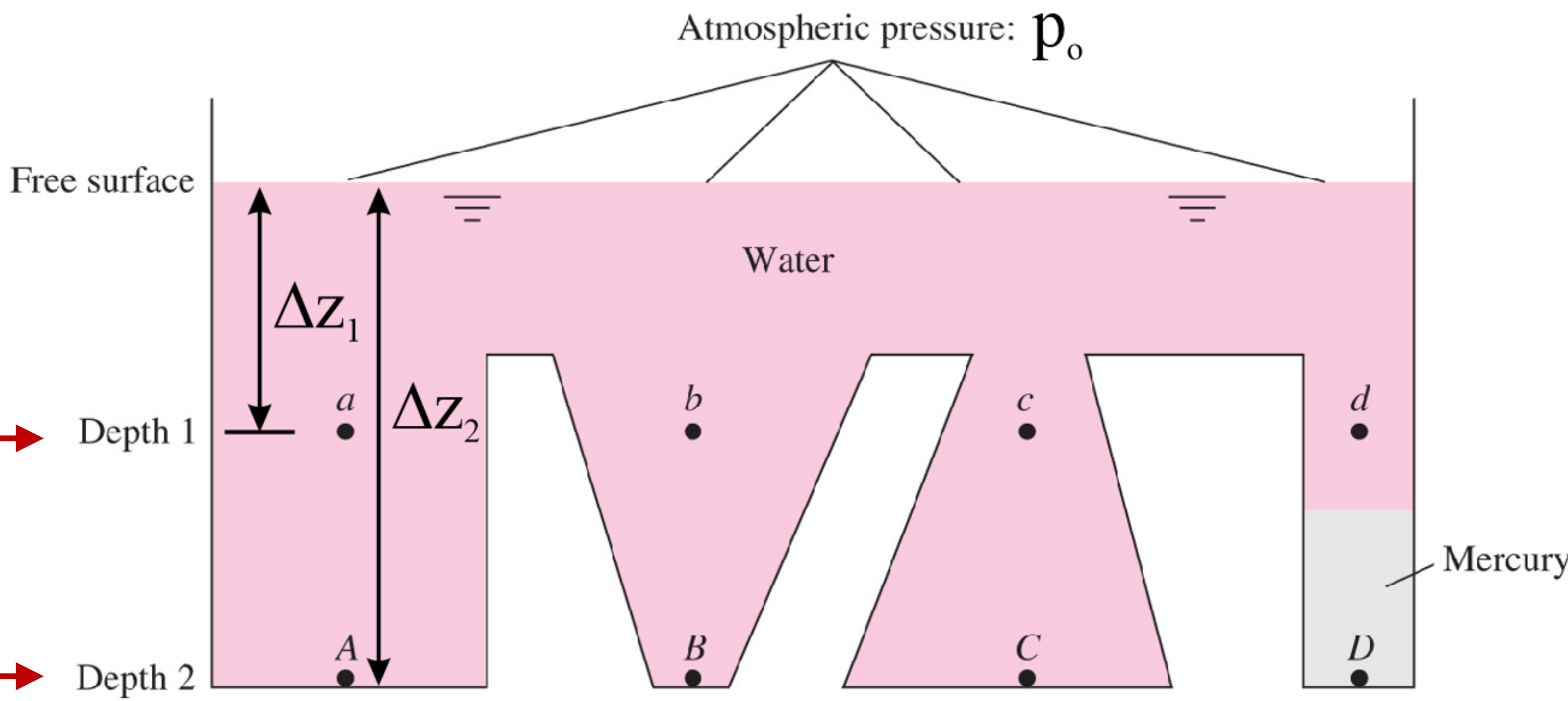
- We have shown that pressure is only a function of depth

Thus, $p_a = p_b = p_c = p_d$ →

$p_a = p_o + \rho_{water}\, g\Delta z_1$

Thus, $p_A = p_B = p_C$ →

$p_A = p_o + \rho_{water}\, g\Delta z_2$



Remember: Same depth in same fluid → same pressure

Question: Is $p_C = p_D$? No! $p_C < p_D$

Hydrostatic Pressure Distribution in a Incompressible Static Fluid

- Physical interpretation can be seen from a simple force balance:

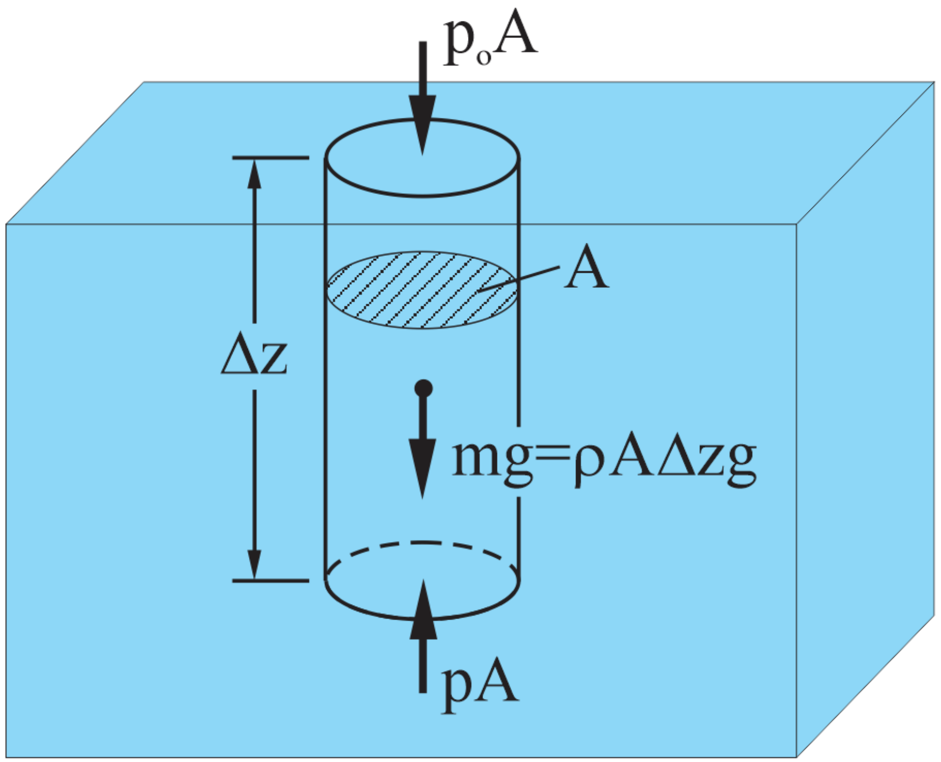
$$\sum F_z = 0$$

$$pA = p_o A + \rho A \Delta z g$$

$$p = p_o + \rho g \Delta z$$

$$p = p_o + \gamma \Delta z$$

(Same Result)



Where $\gamma = \rho g$ is the *specific weight* of the fluid

Key Concept: Pressure a depth Δz is caused by the weight of fluid above that point

Mass of the Earth’s Atmosphere

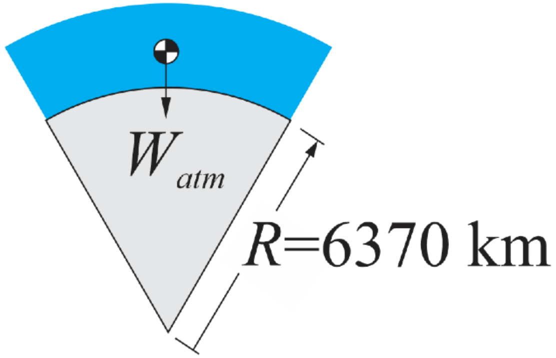
Example

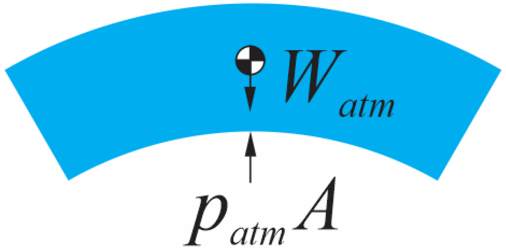
The average pressure at the surface of the Earth (over land and water) is about 100 kPa. The radius of the planet is 6370 km. Using these two facts, calculate the total mass of the Earth’s atmosphere (in kg). Approximate the earth is a perfect sphere, $A=4\pi R^2$.

Solution

The atmospheric pressure is caused by the weight of air:

$$p_{atm}A = W_{atm}$$



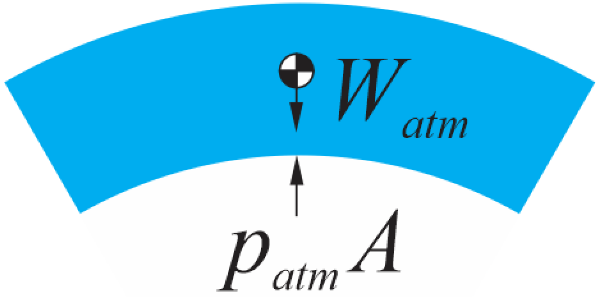


Free Body Diagram

Mass of the Earth's Atmosphere

- The balance of forces gives: $p_{atm}A = W_{atm} = m_{atm}\, g$

- Solving for the total mass of the Earth's atmosphere:



Free Body Diagram

$$m_{atm} = \frac{p_{atm}A}{g} = \frac{p_{atm}\,(4\pi r^2)}{g}$$

$$m_{atm} = \frac{100x10^3 \frac{N}{m^2}\; 4\pi(6370x10^3 m)^2}{9.81\; m/s^2} = 5.18x10^{18}\; kg$$ Ans.

"Google" the answer, NASA website: $5.15x10^{18}\; kg$

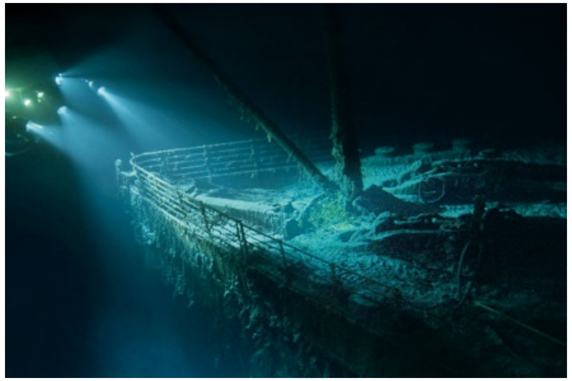
- **Key concept**: Pressure is caused by the weight of fluid

Example: Hydrostatic Pressure

In 1912, the ill-fated RMS Titanic sank to a depth of 3784m. The density of sea water is approximately constant, $\rho \approx 1025\ kg/m^3$.

What is the absolute pressure at this depth?

Example: Hydrostatic Pressure



Solution

$$p = p_o + \rho g \Delta z$$

Depth: $\Delta z = 3784\ m$ Density of sea water: $\rho = 1025\ kg/m^3$

Pressure at the surface ($\Delta z = 0$), i.e. at sea level: $p_o = p_{atm} = 101\ kPa$

$$p = 101x10^3 \frac{N}{m^2} + 1025\ \frac{kg}{m^3}\left(9.81 \frac{m}{s^2}\right) 3784\ m = 3.81x10^7 \frac{N}{m^2}$$

Absolute pressure: $p = 38.1\ MPa$ Ans.

Pressure Distribution in Earth's Atmosphere

- Gases are compressible, $\rho = \rho(p, T)$
- For ideal gases: $\rho = \frac{p}{RT}$ where R=287 J/kgK for air
- Previously, we derived a general expression : $\frac{dp}{dz} = -\rho g$
- Subst. the ideal gas equation of state: $\frac{dp}{dz} = \frac{-pg}{RT}$ (1)
- Temperature decreases in the atmosphere with altitude (z):

$T = T_o - Bz$ where T_o= 15 °C = 288 K and B=0.0065 K/m

B is the rate of decrease of temperature with elevation (about 6.5 °C per km)

Pressure Distribution in Earth's Atmosphere

- Substituting the expression for T(z) into Eq. (1) gives:

$$\int_{p_o}^{p} \frac{dp}{p} = \int_{z=0}^{z} \frac{-\,g\,dz}{R(T_o - Bz)}$$

Integration gives: $p = p_o \left[1 - \frac{Bz}{T_o}\right]^{\frac{g}{RB}}$ where $\frac{g}{RB}$ = 5.26 (air), p_o =101x10^3 Pa, T_o=288K

- This is simple model gives good results in the troposphere, z ≈ 0-11 km

- Key concept: For compressible fluids pressure does not vary linearly with altitude
- See solved Example 2.2(a) in the textbook (Chapter 2)

US Standard Atmosphere

- Includes compressibility effects and decreasing temperature with altitude (-6.5°C/km) i.e., accounts for $\rho=\rho(p,T)$
- Actual local conditions vary
- These standard values are useful for design, e.g. aircraft

Mt. Everest (8848m): p≈31 kPa, T=230K =-43°C

z, m	T, K	p, Pa	ρ, kg/m³
9000	229.66	30,723	0.4661

Table A.6

z, m	T, K	p, Pa	ρ, kg/m³
−500	291.41	107,508	1.2854
0	288.16	101,350	1.2255
500	284.91	95,480	1.1677
1000	281.66	89,889	1.1120
1500	278.41	84,565	1.0583
2000	275.16	79,500	1.0067
2500	271.91	74,684	0.9570
3000	268.66	70,107	0.9092
3500	265.41	65,759	0.8633
4000	262.16	61,633	0.8191
4500	258.91	57,718	0.7768
5000	255.66	54,008	0.7361
5500	252.41	50,493	0.6970
6000	249.16	47,166	0.6596
6500	245.91	44,018	0.6237
7000	242.66	41,043	0.5893
7500	239.41	38,233	0.5564
8000	236.16	35,581	0.5250
8500	232.91	33,080	0.4949
9000	229.66	30,723	0.4661
9500	226.41	28,504	0.4387
10,000	223.16	26,416	0.4125
20,000	216.66	5467	0.0879
30,000	226.5	1197	0.0184
40,000	250.4	287	0.0040
50,000	270.7	80	0.0010
60,000	255.7	22	0.0003
70,000	219.7	6	0.0001

Pressure Measurement

Absolute Pressure

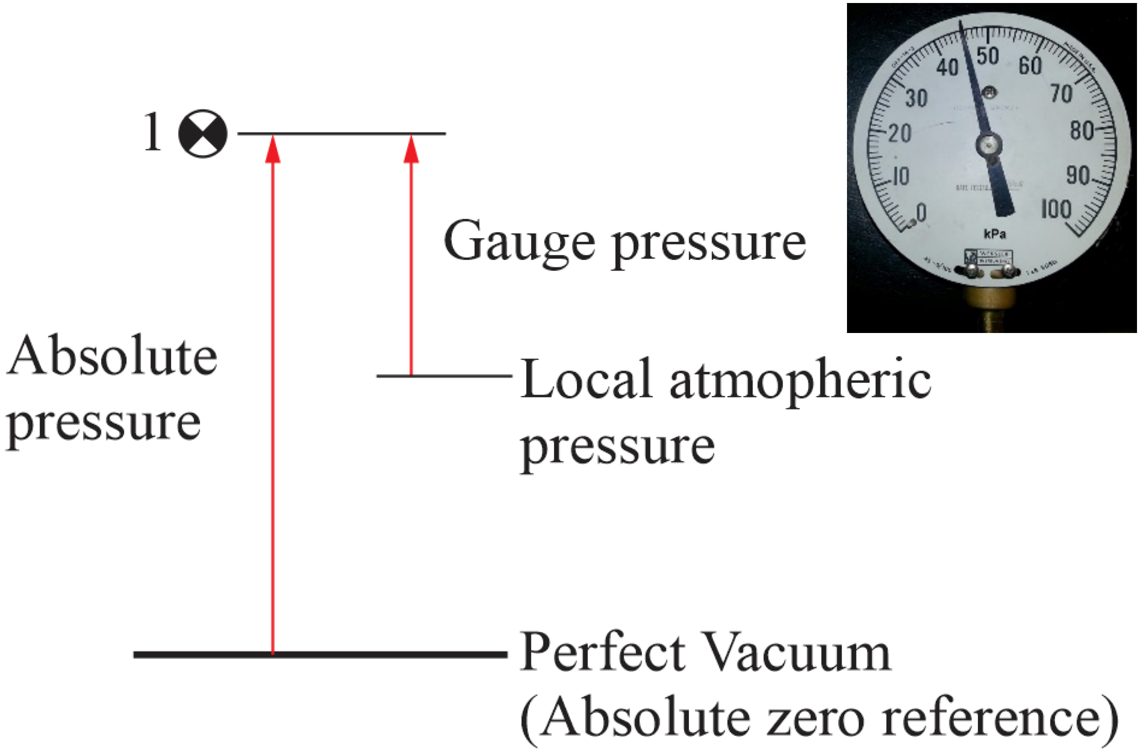
- Absolute pressure is measured relative to a perfect vacuum:

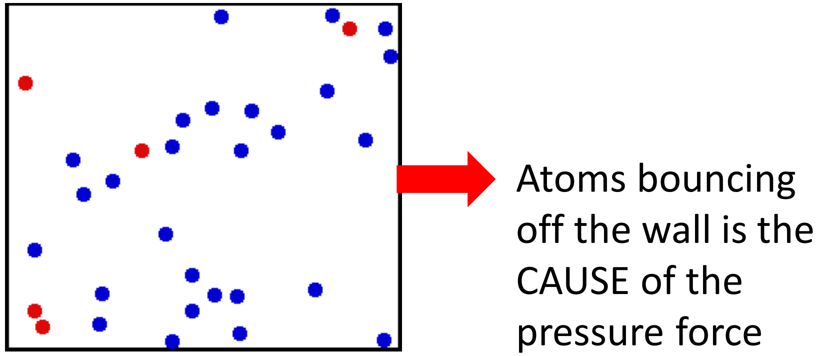
$$p = 0 \;\; Pa \; (a)$$

- Absolute pressures are always positive
- e.g. p_{atm}=101.3 kPa is an absolute pressure

Gauge Pressure

- Gauge pressure is measured relative to local atmospheric pressure
- Measured by a typical pressure gauges

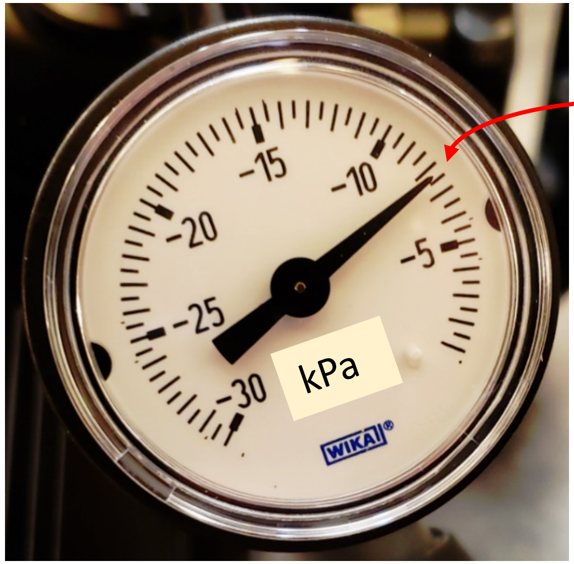




Atoms in a gas

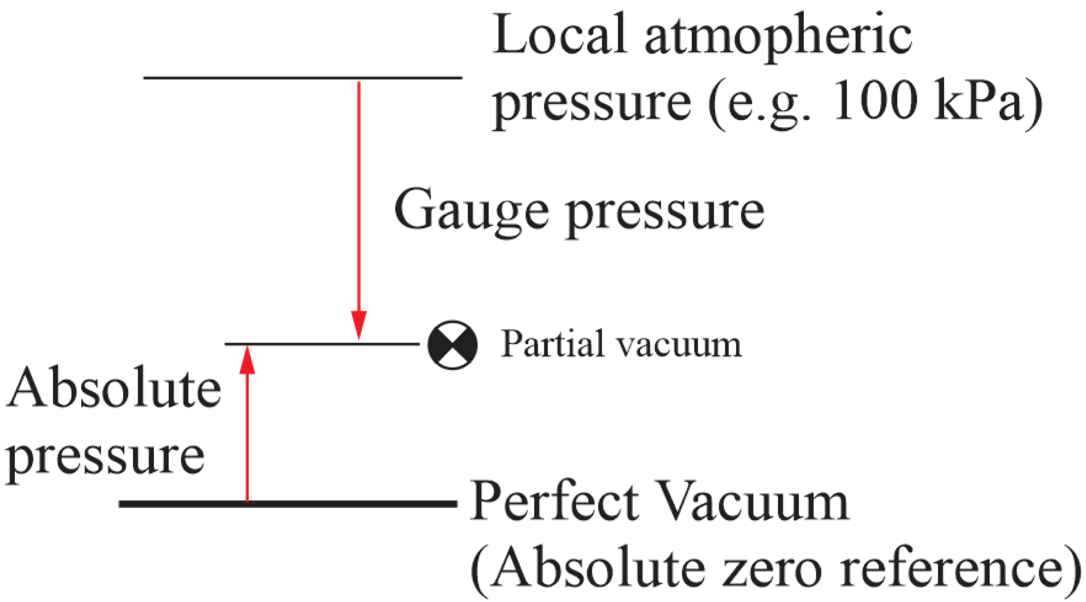
Gauge Pressure

- Gauge pressure can be positive or negative
- A negative gauge pressure is also referred to vacuum pressure



p=-7.5 kPa (g)

What does a negative pressure mean?



Concept: Pressure in an "Empty" Spray Paint Can

Imagine you are spray painting with a can. The local atmospheric pressure is 99.8 kPa. The can runs out and you cannot spray any more paint.

- What is the absolute pressure inside the can?

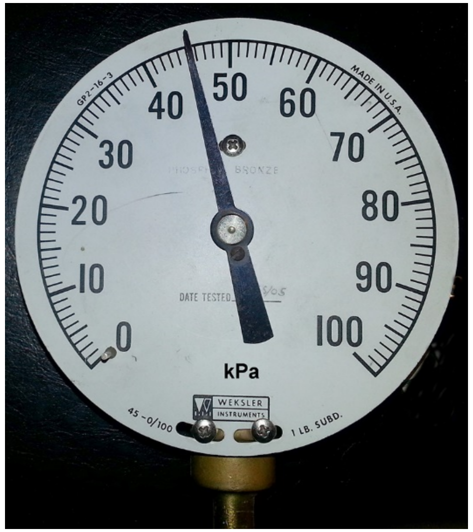
Ans: $p = 99.8\ kPa\ (a)$

- What is the gauge pressure inside the can?

Ans: $p = 0\ kPa\ (g)$

An "empty" can contains propellant at local atmospheric pressure

Absolute and Gauge Pressure



Example: Suppose you want to calculate the density of air in a pressurized tank. The gauge on a tank reads 45 kPa and the local atmospheric pressure is $p_{atm} = 99\ kPa$. What pressure would you use in the ideal gas equation of state, $\rho = p/(RT)$?

Answer:

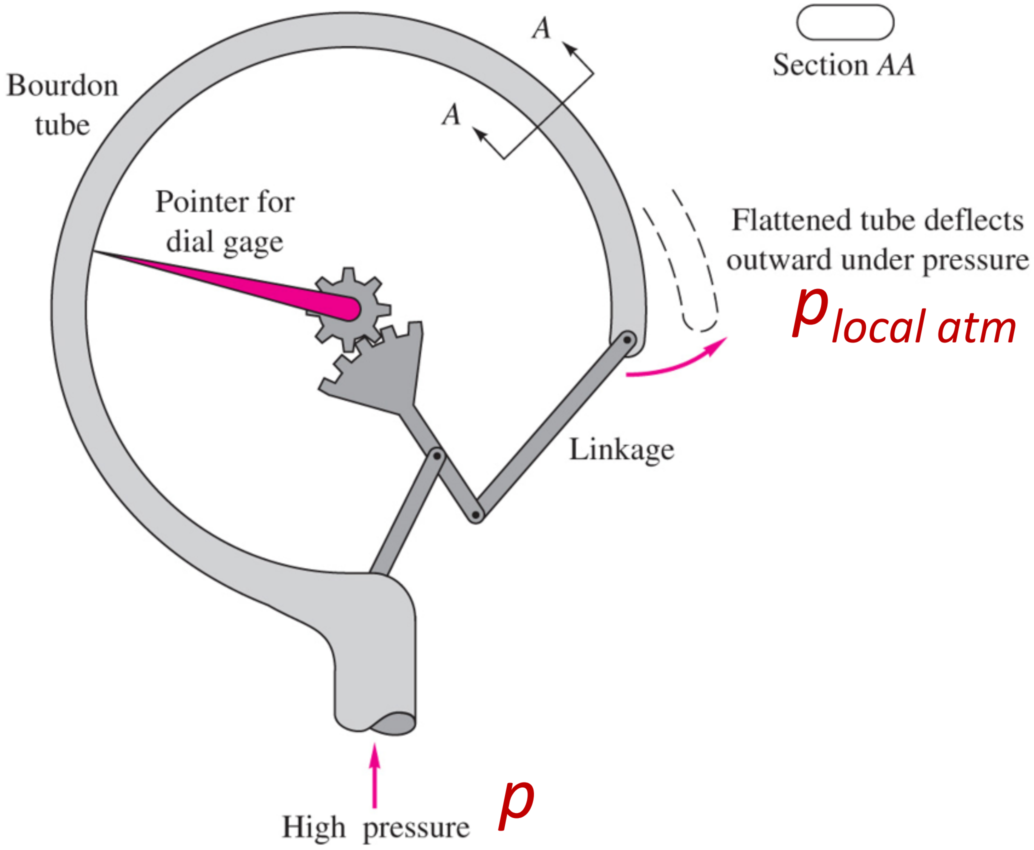
Absolute pressure must be used in the ideal gas equation:

$$p_{abs} = p_{gauge} + p_{atm} = 45\ kPa(g) + 99\ kPa = 144\ kPa(a)$$

If you used the gauge pressure the calculated air density, you would be ***wrong*** by a factor of ~ 3!

The Bourdon Tube Pressure Gauge

- Why do gauges measure relative to local atmospheric pressure?
- The pressure difference across the tube that causes it to bend is $p-p_{atm.}$ Thus, they measure *gauge pressure*



Bourdon Tube Pressure Gauge

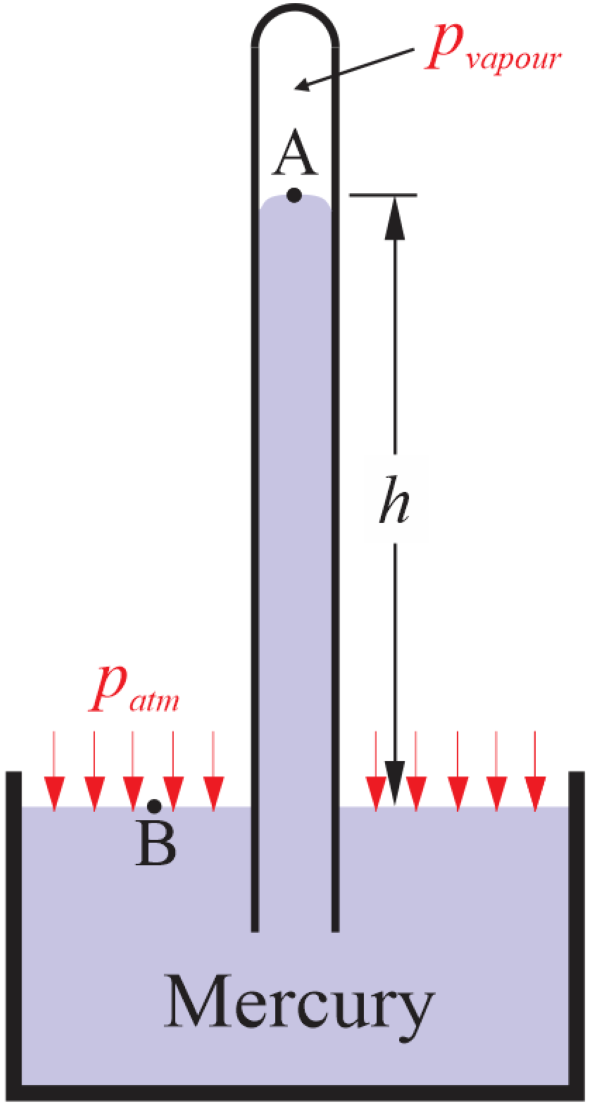
Mercury Barometer

- Used to measure atmospheric pressure (since 1600s)
- Glass tube filled with mercury and turned upside down
- Simple application of our hydrostatic formula:

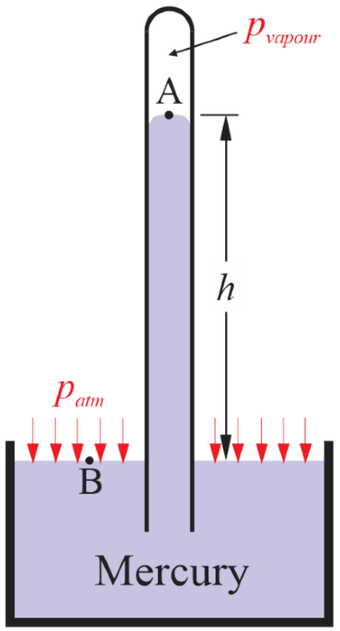
$$p_B - p_A = \rho g h = \gamma h$$

$$p_A = p_{vapor} \approx 0 \qquad (p_{vapour} = 0.0016\, kPa \text{ at } 20^{o}\text{C})$$

So, $p_{atm} = p_B = \gamma h$



Mercury Barometer



- Specific gravity of mercury: SG=13.55

$$\gamma = \rho g = SG\, \rho_{water}\, g = 13.55\left(1000\,\frac{kg}{m^3}\right)9.81\,\frac{m}{s^2} = 132900\,N/m^3$$

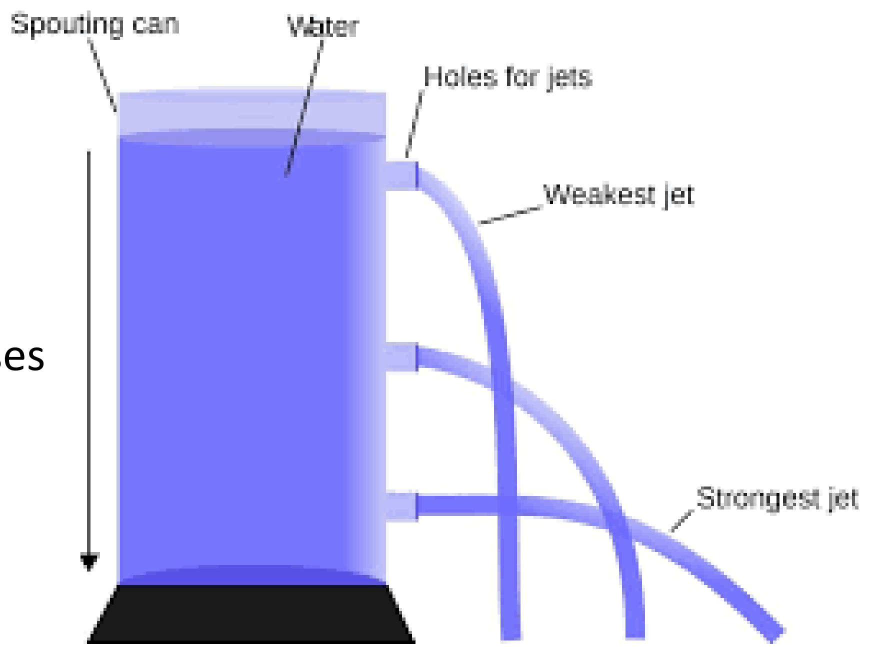
- Standard atmo. pressure is h=760 mmHg

Thus, $p_{atm} = \gamma h = 132900\,\frac{N}{m^3}(0.760m) = 101\times 10^3 Pa = 101\,kPa$

- This is an ***absolute*** pressure (not gauge pressure)
- Pressure is often reported in mmHg

1 Torr= 1 mmHg = 133 Pa (after Torricelli)

Pressure increases with depth



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END NOTES

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