Last Updated: May 2021

MEC516/BME516
Fluid Mechanics I

Chapter 1
Recommended Problem Solutions

Caution: Reading solutions can be deceiving. Solutions that look obvious at a glance can be difficult to reproduce later. Consult these solutions only after making an honest independent attempt at each problem.

1. Kinematic viscosity is $\nu = \frac{\mu}{\rho}$

Using specific gravity: $SG = \frac{\rho}{\rho_{w\ (at\ 4^oC)}}$ $\rho = 0.870\left(1000\frac{kg}{m^3}\right) = 870\frac{kg}{m^3}$

$$\nu = \frac{0.104\frac{kg}{ms}}{870\frac{kg}{m^3}} = 1.19x10^{-4}\frac{m^2}{s}$$

2. Consider a volume ∀ of beer. The mass of ethanol and the mass of water in this volume are:

$$m_e = 0.05\ \forall\ \rho_e \quad m_w = 0.95\ \forall\ \rho_w$$

The density of the beer is: $\rho = \frac{m}{\forall} = \frac{m_e+m_w}{\forall}$

So, the specific gravity of the beer can be expressed as:

$$SG = \frac{\rho}{\rho_{w\ (at\ 4^oC)}} = \frac{m_e + m_w}{\forall\ \rho_{w\ (at\ 4^oC)}} = \frac{0.05\ \forall\ \rho_e + 0.95\ \forall\ \rho_w}{\forall\ \rho_{w\ (at\ 4^oC)}}$$

The volume ∀ cancels. From Table A.3, the density of ethanol at 20ºC is $\rho_e = 789\ kg/m^3$.

$$SG = \frac{0.05\ \rho_e + 0.95\ \rho_w}{\rho_{w\ (at\ 4^oC)}} = \frac{0.05\ (789) + 0.95(998)}{1000} = 0.988$$

3. Consider any arbitrary 5% mixture by mass: For example, consider 950kg of water mixed with 50kg of ethanol.

The volume of the water is: $\forall_w = \frac{m_w}{\rho_w} = \frac{950\ kg}{998\ kg/m^3} = 0.95190\ m^3$

The volume of the ethanol is: $\forall_e = \frac{m_e}{\rho_e} = \frac{50\ kg}{789\ kg/m^3} = 0.06337\ m^3$

The density of the beer is: $\rho = \frac{m}{\forall} = \frac{m_e+m_w}{\forall_e+\forall_w} = \frac{50\ kg+950\ kg}{0.06337\ m^3+0.95190\ m^3} = 985.0\ kg/m^3$

$$SG = \frac{\rho}{\rho_{w\ (at\ 4^oC)}} = \frac{985.0\ kg/m^3}{1000\ kg/m^3} = 0.985$$

The specific gravity for a 5% mixture by mass is slightly lower than for a 5% mixture by volume. The difference is small. However, that will not always be the case.

4. The constants 3π and 9π/16 have no dimensions, i.e. {1}. Thus, we need to show that:

$$\{F\} = \{\mu\}\{D\}\{V\} + \{\rho\}\{V\}^2\{D\}^2$$

From Newton's second law: $\{F\} = \{m\}\{a\} = \left\{\frac{ML}{T^2}\right\}$

$$\left\{\frac{ML}{T^2}\right\} = \left\{\frac{M}{LT}\right\}\{L\}\left\{\frac{L}{T}\right\} + \left\{\frac{M}{L^3}\right\}\left\{\frac{L}{T}\right\}^2\{L\}^2$$

Simplifying,

$$\left\{\frac{ML}{T^2}\right\} = \left\{\frac{ML}{T^2}\right\} + \left\{\frac{ML}{T^2}\right\}$$

The equation is dimensionally homogeneous: All terms have the same dimensions. You will need this skill for Chapter 5 (Dimensional Analysis).

5. Volume flow rate Q is volume per unit time and has dimensions of $\{L^3/T\}$:

$$\left\{\frac{L^3}{T}\right\} = \frac{\{1\}\{L\}^n\left\{\frac{F}{L^2}\right\}}{\left\{\frac{M}{LT}\right\}\{L\}} = \frac{\{1\}\{L\}^n\left\{\frac{ML}{T^2L^2}\right\}}{\left\{\frac{M}{LT}\right\}\{L\}} = \frac{\{L\}^{n-1}}{\{T\}}$$

To be dimensionally homogeneous, the exponents of {L} must be the same on both sides of the equation:

$$3 = n - 1, \qquad n = 4$$

6. Recall from basic calculus that differentials represent differences in variables: $\partial y = \Delta y$ as $\Delta y \to 0$. For this reason, ∂y has dimensions of $\{L\}$ and ∂u has dimensions of $\{L/T\}$. The second derivative term has the following dimensions:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left\{\frac{1}{L}\right\}\left\{\frac{L}{TL}\right\} = \left\{\frac{1}{LT}\right\}$$

Using this result in the x-momentum equation:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \rho g_x$$

$$\left\{\frac{M}{L^3}\right\}\left\{\frac{L}{TT}\right\} + \left\{\frac{M}{L^3}\right\}\left\{\frac{L}{T}\right\}\left\{\frac{L}{TL}\right\} + \left\{\frac{M}{L^3}\right\}\left\{\frac{L}{T}\right\}\left\{\frac{L}{TL}\right\} = \left\{\frac{ML}{T^2L^2L}\right\} + \left\{\frac{M}{LT}\right\}\left\{\frac{1}{TL}\right\} + \left\{\frac{M}{L^3}\right\}\left\{\frac{L}{T^2}\right\}$$

Simplifying:

$$\left\{\frac{M}{T^2L^2}\right\} + \left\{\frac{M}{T^2L^2}\right\} + \left\{\frac{M}{T^2L^2}\right\} = \left\{\frac{M}{T^2L^2}\right\} + \left\{\frac{M}{T^2L^2}\right\} + \left\{\frac{M}{T^2L^2}\right\}$$

Thus, the equation is homogeneous.

7. Table A.4 gives the gas constant for carbon monoxide: $R_{CO} = 297\, J/(kgK)$. Using the ideal gas equation of state to calculate the density:

$$\rho = \frac{p}{RT} = \frac{0.40x10^6 \frac{N}{m^2}}{297 \frac{Nm}{kgK}(45+273)K} = 4.233\, \frac{kg}{m^3}$$

Noting that 1m^3 =1000 litres, the tank volume is:

$$\forall = \frac{m}{\rho} = \frac{5.0\, kg}{4.233\, kg/m^3} = 1.18\, m^3 = 1180\, l$$

8. The cylinder is falling a constant (terminal) velocity, i.e. the acceleration is zero. So, the forces in the vertical direction must be balanced. See the sketch of the problem below. Oil fills the gap between the cylinder and sleeve. The cylinder's weight W is balanced by the upward viscous shear force on the outer surface area of the cylinder. Table A.3 gives the viscosity of SAE 50W oil at 20°C: μ=0.860 Ns/m^2.

$$\sum F_{vertical} = ma = 0$$

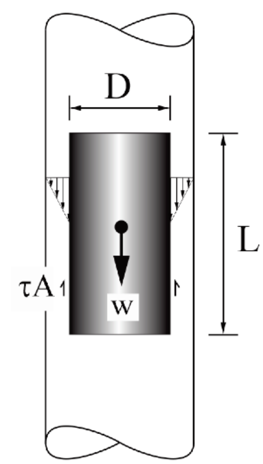
$$\tau A = W$$

Applying Newton's law of viscosity:

$$\tau A = \mu \frac{du}{dr} A = W$$

$$\mu \frac{\Delta u}{\Delta r}(\pi DL) = W$$

$$0.86 \frac{Ns}{m^2}\left(\frac{V}{0.0002m}\right)\pi(0.06m)(0.40m) = 30\, N$$



Solving for the terminal velocity:

$$V = 0.0925 \frac{m}{s}$$

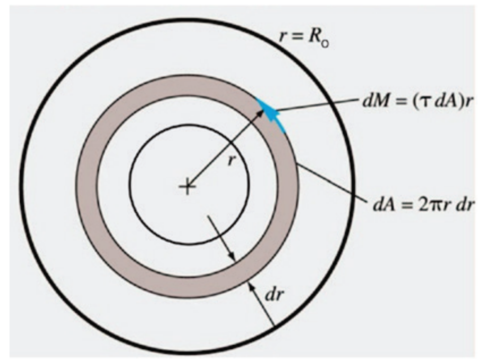
Note: A common error is to use Δr=0.4mm. The radial clearance is half the difference in the diameters, Δr=0.2 mm. Also, note that the cylinder and sleeve contact areas are approximately equal, $D_s \approx D$.

9. See the diagram below. The local surface velocity (rΩ) varies with r. Hence, the local viscous shear moment varies with radius r and must be integrated over the disk. Note that the differential moment is the local shear force τdA times the moment arm *r*:

$$M = \int_{R_i}^{R_o} dM = \int_{R_i}^{R_o} r\,\tau\,dA \quad where\ \ dA = 2\pi r dr$$

The local shear stress is:

$$\tau = \mu \frac{du}{dy} = \mu \frac{\Omega\, r}{h}$$



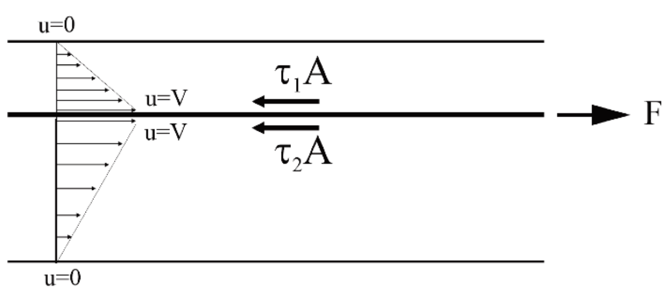
Making the substitutions, the total moment M is:

$$M = \int_{R_i}^{R_o} r\,\mu \frac{\Omega\, r}{h}(2\pi r)dr = \frac{2\pi\mu\Omega}{h}\int_{R_i}^{R_o} r^3\,dr = \frac{2\pi\mu\Omega}{h}\left[\frac{r^4}{4}\right]_{R_i}^{R_o} = \frac{\pi\mu\Omega}{2h}\left(R_o^4 - R_i^4\right)$$

An Aside: This is not required by the problem. For added confidence (as a check that a parameter wasn't accidentally dropped) one can confirm that the expression is dimensionally homogenous:

$$\left\{\frac{ML^2}{T^2}\right\} = \left\{\frac{M}{LT}\right\}\left\{\frac{1}{T}\right\}\left\{\frac{1}{L}\right\}\{L^4\} = \left\{\frac{ML^2}{T^2}\right\}$$

10. The centre plate is moving at constant velocity, i.e. $\sum F = ma = 0$.



The force *F* is balanced by the two viscous shear forces:

$$F = \tau_1 A + \tau_2 A = \mu_1 \left.\frac{du}{dy}\right|_1 A + \mu_2 \left.\frac{du}{dy}\right|_2 A = \left(\mu_1 \frac{V}{h_1} A + \mu_2 \frac{V}{h_2} A\right)$$

$$F = \left(\frac{\mu_1}{h_1} + \frac{\mu_2}{h_2}\right) VA$$

11. The type of flow depends upon the *Reynolds number*. The properties of liquid water at 50ºC from Table A.1: ρ=988 kg/m^3, μ=0.548x10^{-3} kg/(ms).

The *Reynolds number* for the domestic hot water pipe flow is:

$$Re = \frac{\rho V D_i}{\mu} = \frac{988\frac{kg}{m^3}\left(0.5\frac{m}{s}\right)0.013m}{0.548x10^{-3}\frac{kg}{ms}} = 11{,}700 \ \ (Turbulent\ flow)$$

It has been observed experimentally that the transition from laminar to turbulent flow in a round pipe starts at Re≈2,300. This Reynolds number is sufficiently high to have fully turbulent flow in the pipe. (Most real-world flows are turbulent.)

12. The shear stress at the wall (y=0) is:

$$\tau_{wall} = \mu \frac{du}{dy}\bigg|_{y=0} = \mu\,\frac{\pi U_\infty}{2\delta}\cos\left(\frac{\pi y}{2\delta}\right)\bigg|_{y=0} = \mu\,\frac{\pi U_\infty}{2\delta}$$

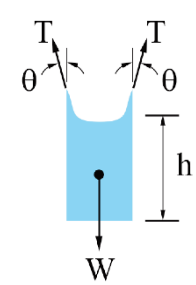
For air at 1atm and -40ºC, Table A.2: ρ=1.52 kg/m^3, μ=1.51x10^{-5} Ns/m^2.

$$\tau_{wall} = \mu\,\frac{\pi U_\infty}{2\delta} = \left(1.51x10^{-5}\,\frac{Ns}{m^2}\right)\frac{\pi\left(5.2\frac{m}{s}\right)}{2(0.0045m)} = 0.0274\,\frac{N}{m^2} = 0.0274\ Pa \rightarrow$$

The viscous shear force on the surface acts in the direction of flow (to the right).

13. Surface tension of a water-air interface at 60ºC, Table A.5: ϒ=0.0662 N/m. Table A.1: ρ=983 kg/m^3. Surface tension is the force per unit length at the interface where the water meets the glass. Thus, $T = \Upsilon L$, where L is the length of the interface. A free body diagram of the water column is shown in the sketch below. For static equilibrium, the surface tension on both sides supports the weight of the water column:

$$\sum F_{vertical} = 0 \quad \rightarrow \qquad 2T\,cos\theta = 2\Upsilon L\,cos\theta = W$$



Free Body Diagram

The weight of the water column is:

$$W = mg = \rho \forall g = \rho\,(hLd)g$$

Making the substitution:

$$2\Upsilon L\,cos\theta = \rho(hLd)g$$

$$h = \frac{2\Upsilon\,cos\theta}{\rho d g} = \frac{2\left(0.0662\frac{N}{m}\right)\cos(0)}{983\frac{kg}{m^3}(0.0005m)9.81\frac{m}{s^2}} = 0.0275m = 27.5\ mm$$

A substantial distance!

Note: In Chapter 2 we will use the height of a liquid column to measure pressure. For this application it is necessary to use sufficiently large diameter tubes (typically $D_i > 5\ mm$) to ensure that capillary effects are negligible.